COV886 Special Module in Algorithms: Computational Social Choice

Lecture 6

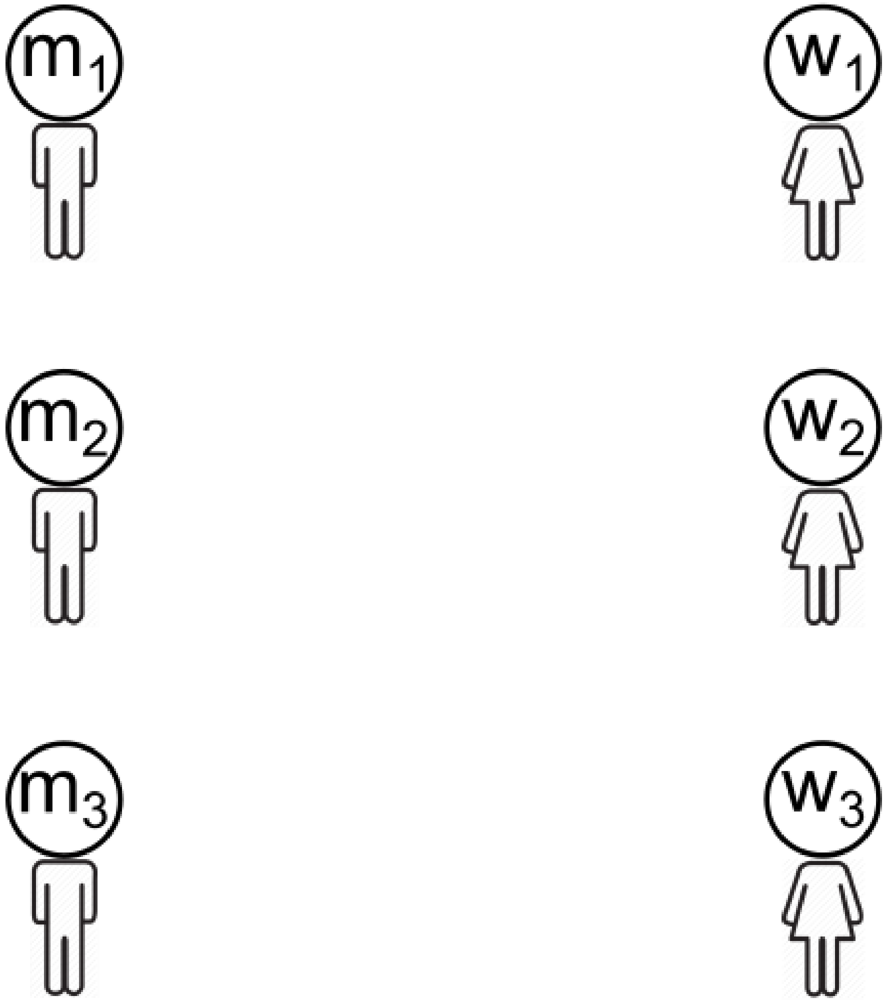
Stable Matchings

Feb 19, 2022 | Rohit Vaish

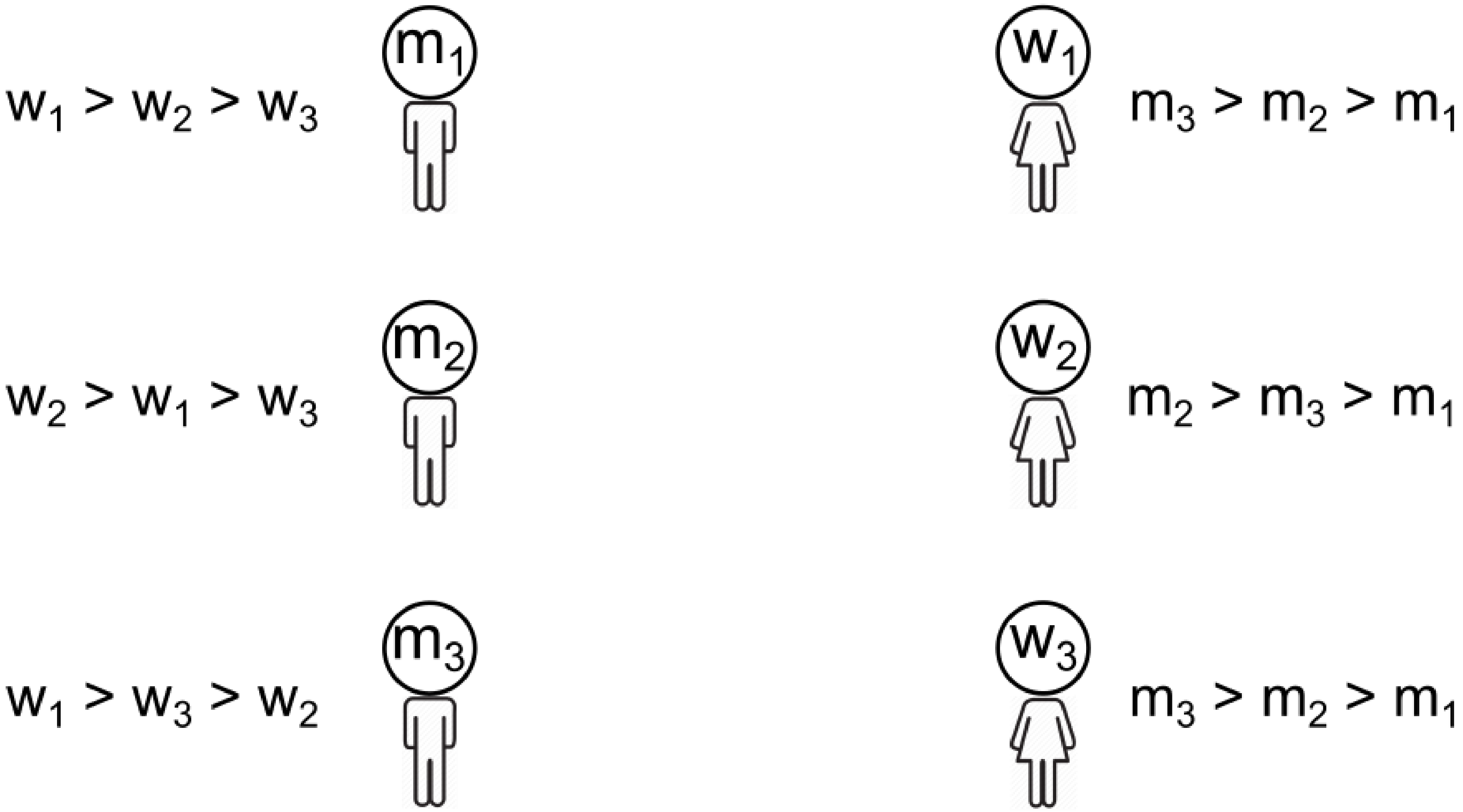
Reminder about starting recording

Stable Matching Problem

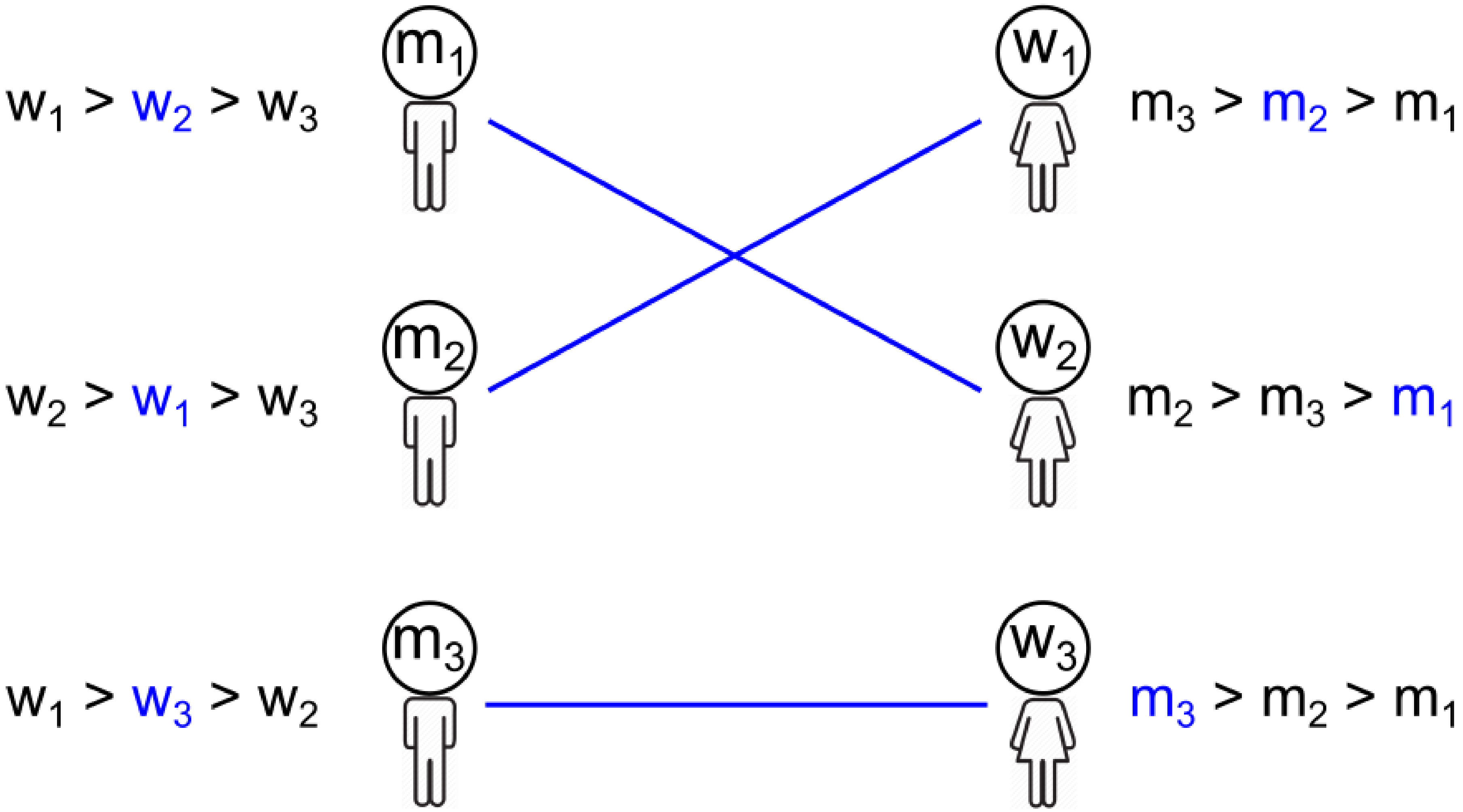
Stable Matching Problem



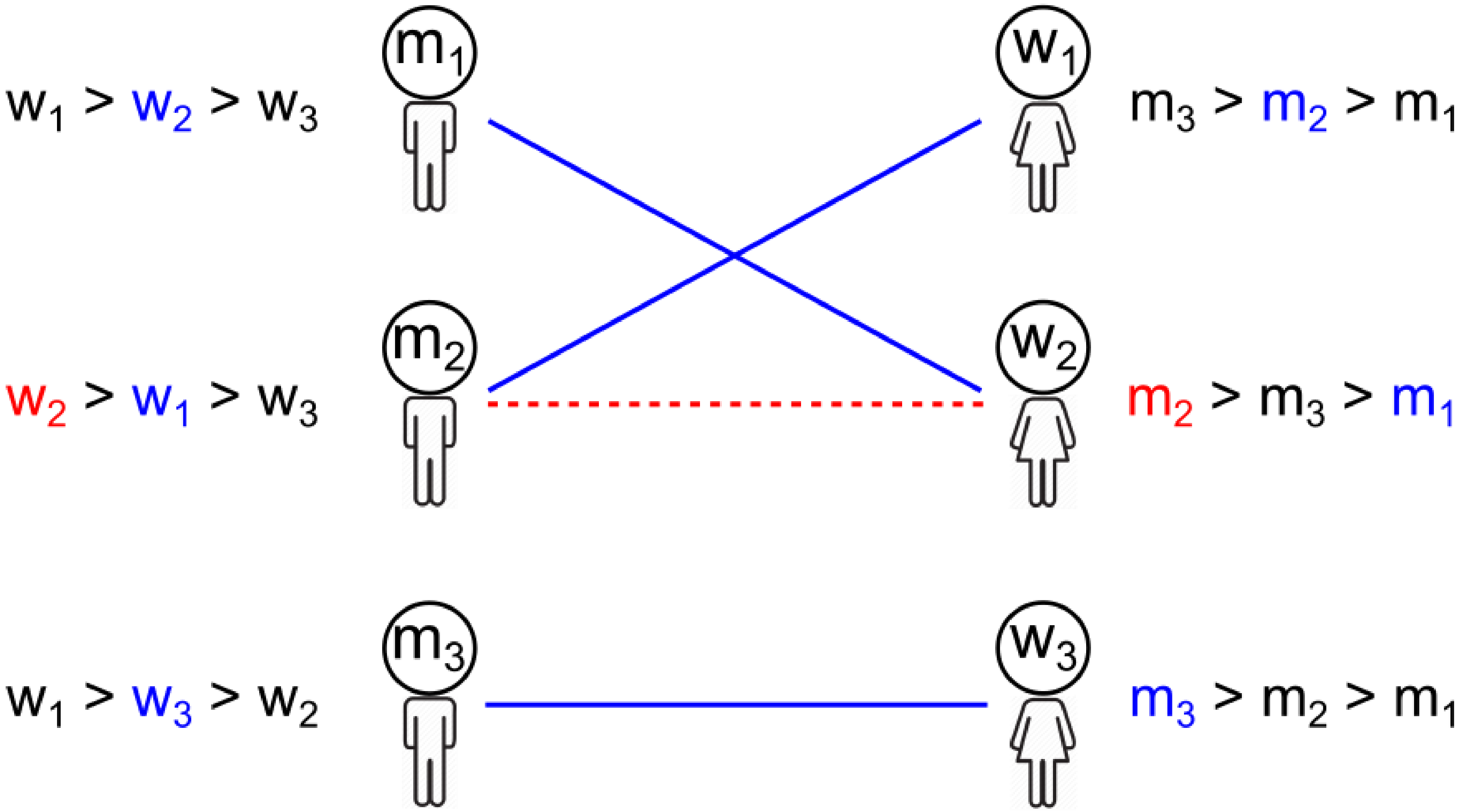
Stable Matching Problem



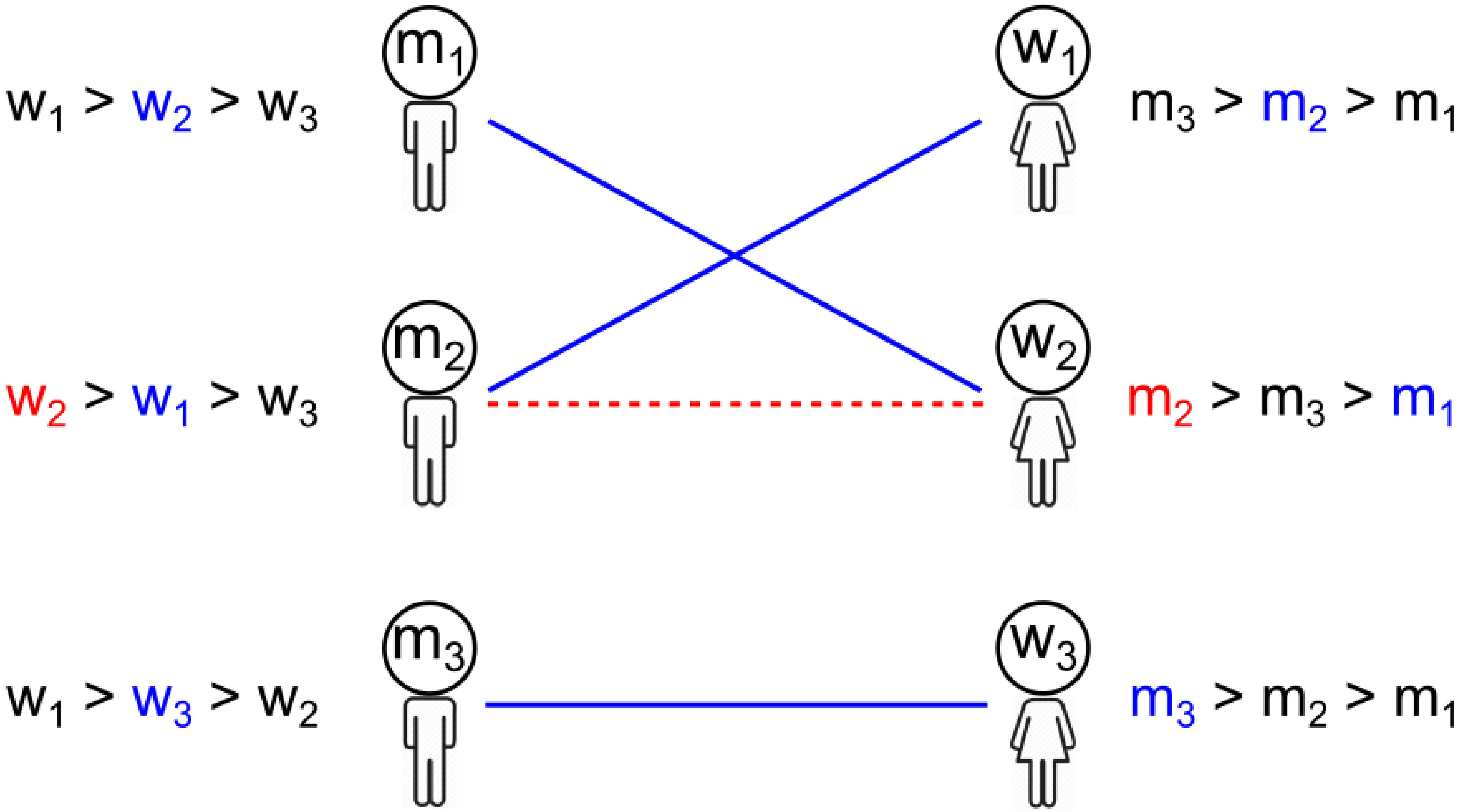
Stable Matching Problem



Stable Matching Problem

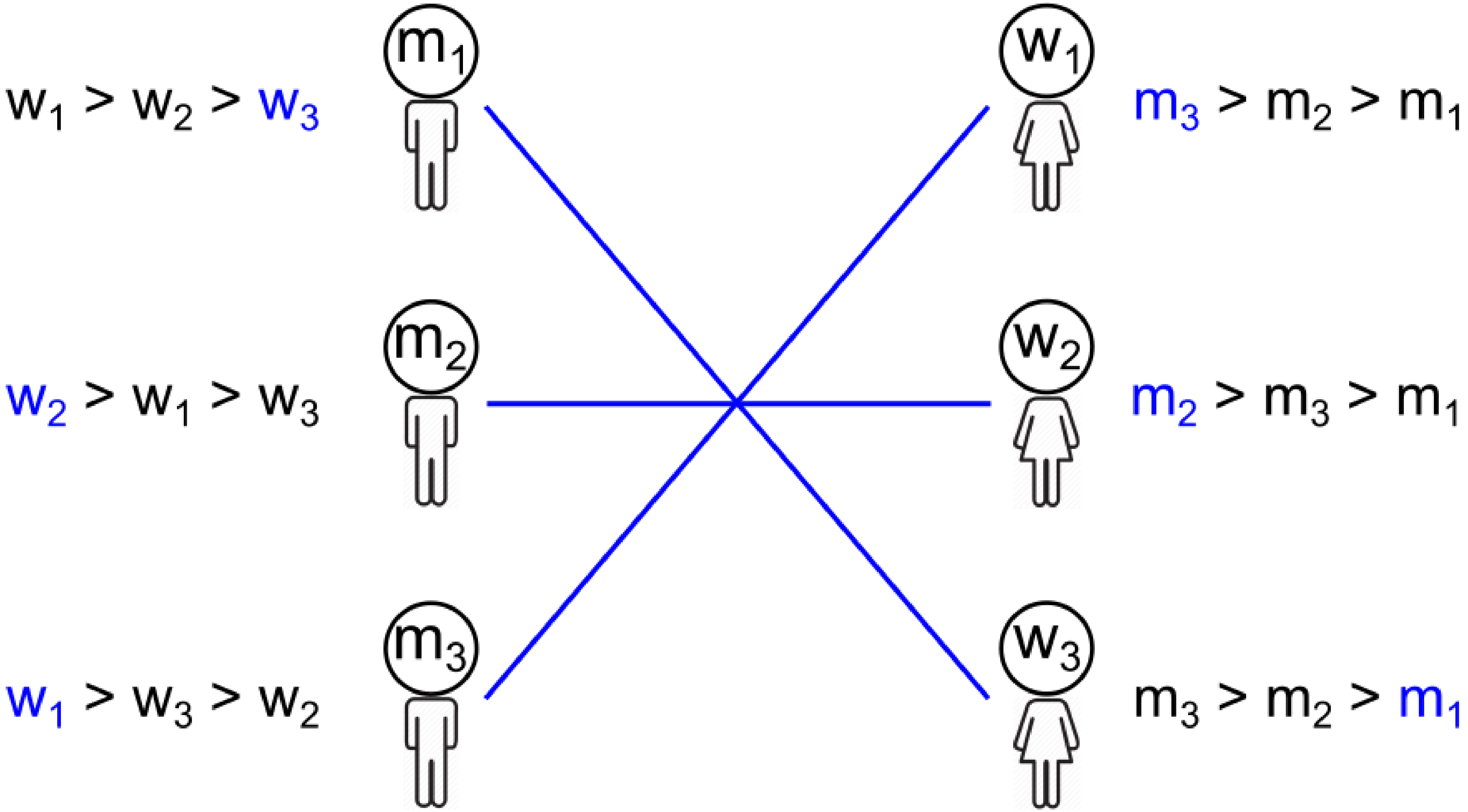


Stable Matching Problem



A matching is stable if there is no blocking pair.

Stable Matching Problem



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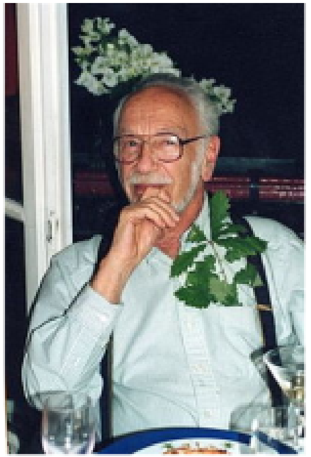


COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9–15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

Deferred-Acceptance Algorithm

Deferred-Acceptance Algorithm

$w_1 > w_2 > w_3 > w_4$	m_1	w_1	$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$	m_2	w_2	$m_4 > m_2 > m_3 > m_1$
$w_1 > w_2 > w_3 > w_4$	m_3	w_3	$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$	m_4	w_4	$m_1 > m_2 > m_3 > m_4$

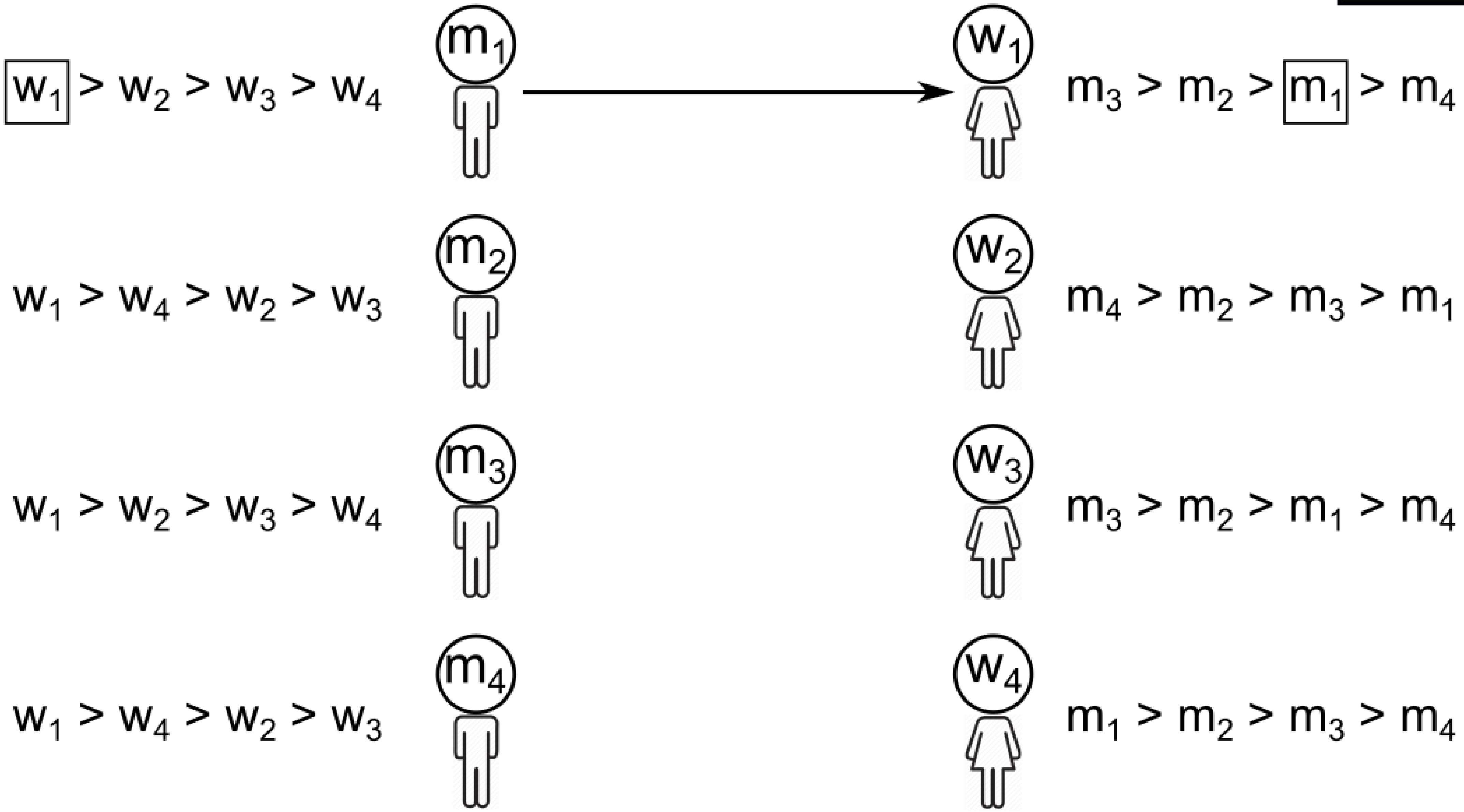
Deferred-Acceptance Algorithm

Round 1

$w_1 > w_2 > w_3 > w_4$	m_1	w_1	$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$	m_2	w_2	$m_4 > m_2 > m_3 > m_1$
$w_1 > w_2 > w_3 > w_4$	m_3	w_3	$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$	m_4	w_4	$m_1 > m_2 > m_3 > m_4$

Deferred-Acceptance Algorithm

Round 1



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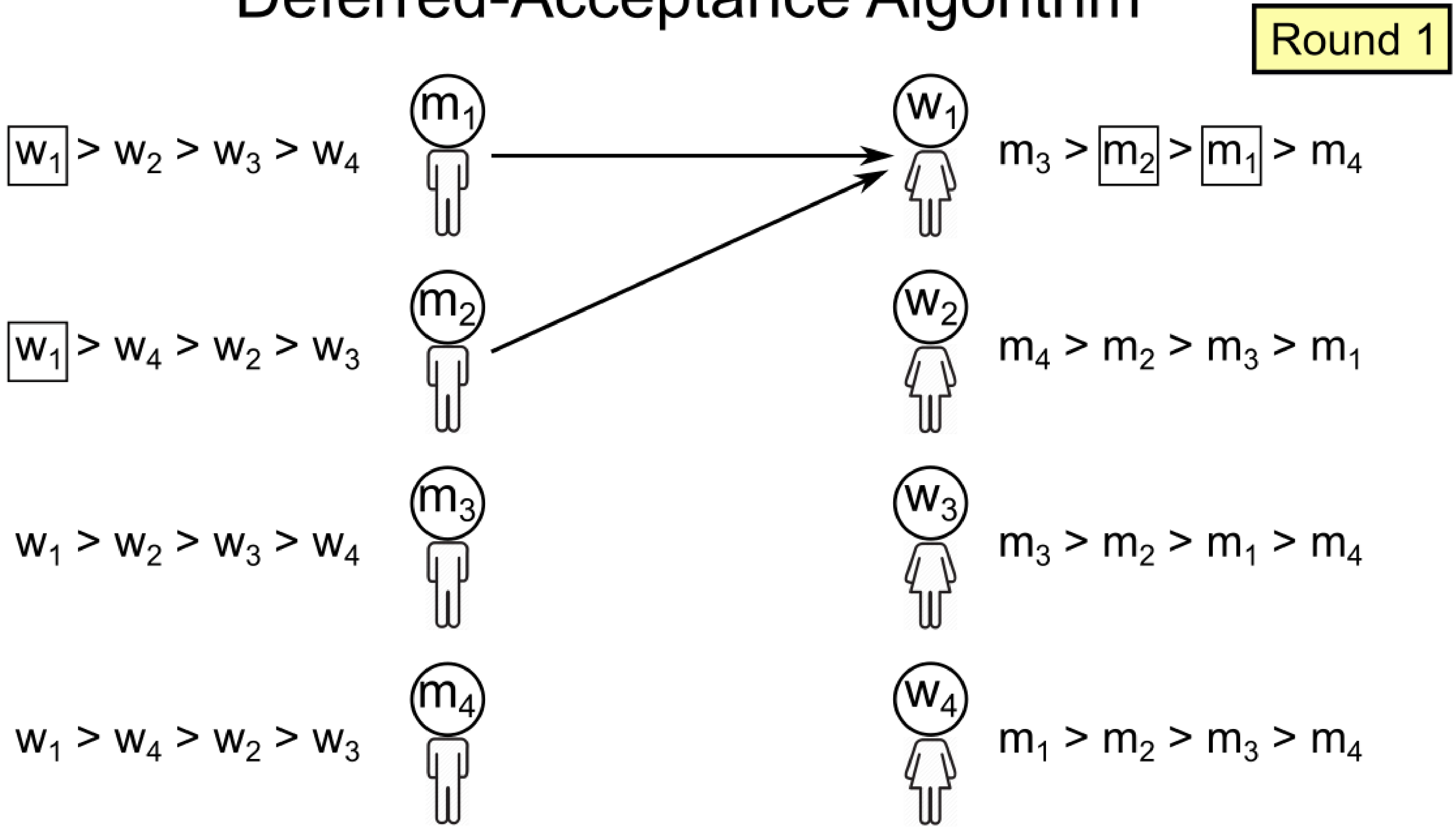

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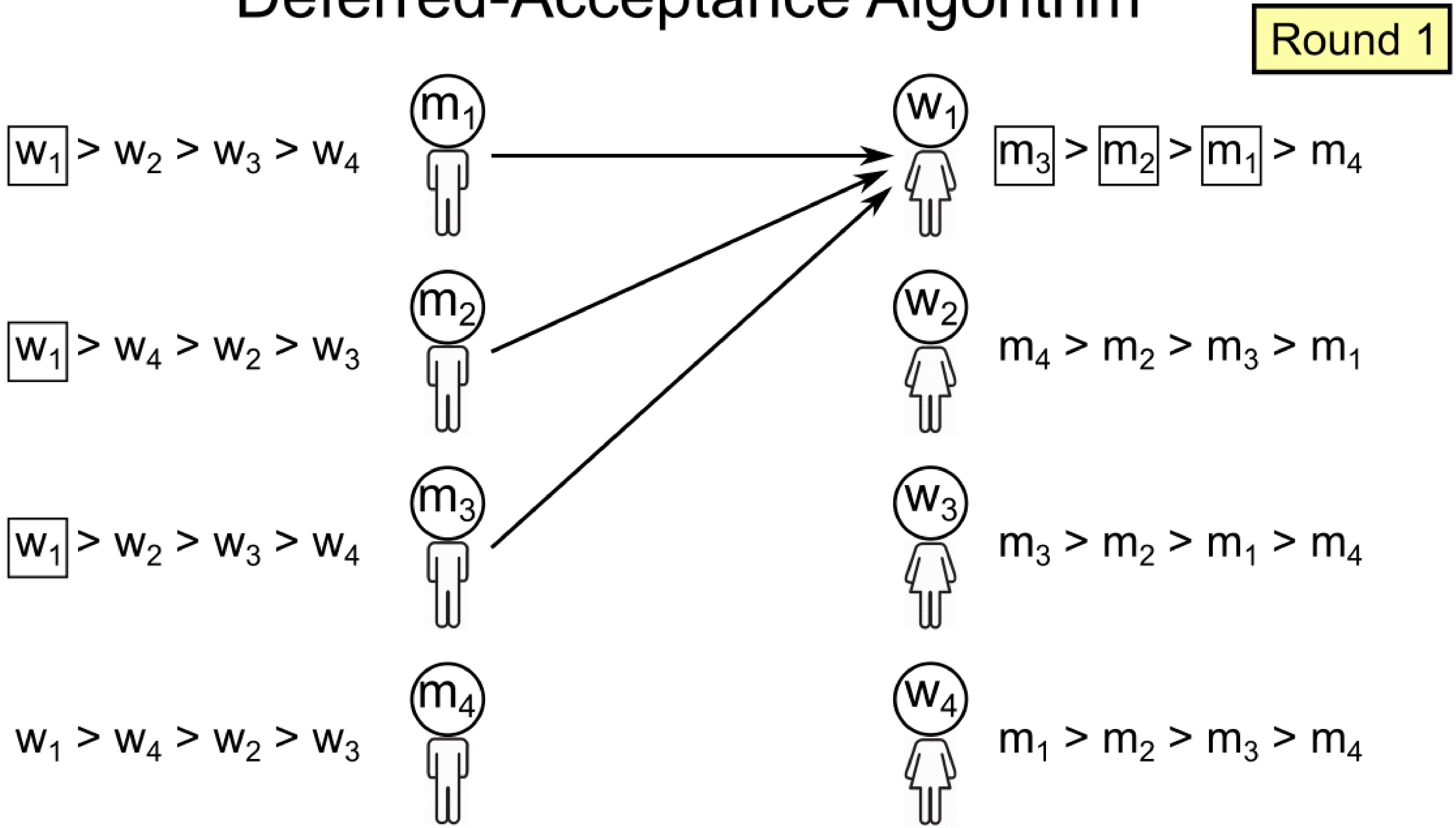

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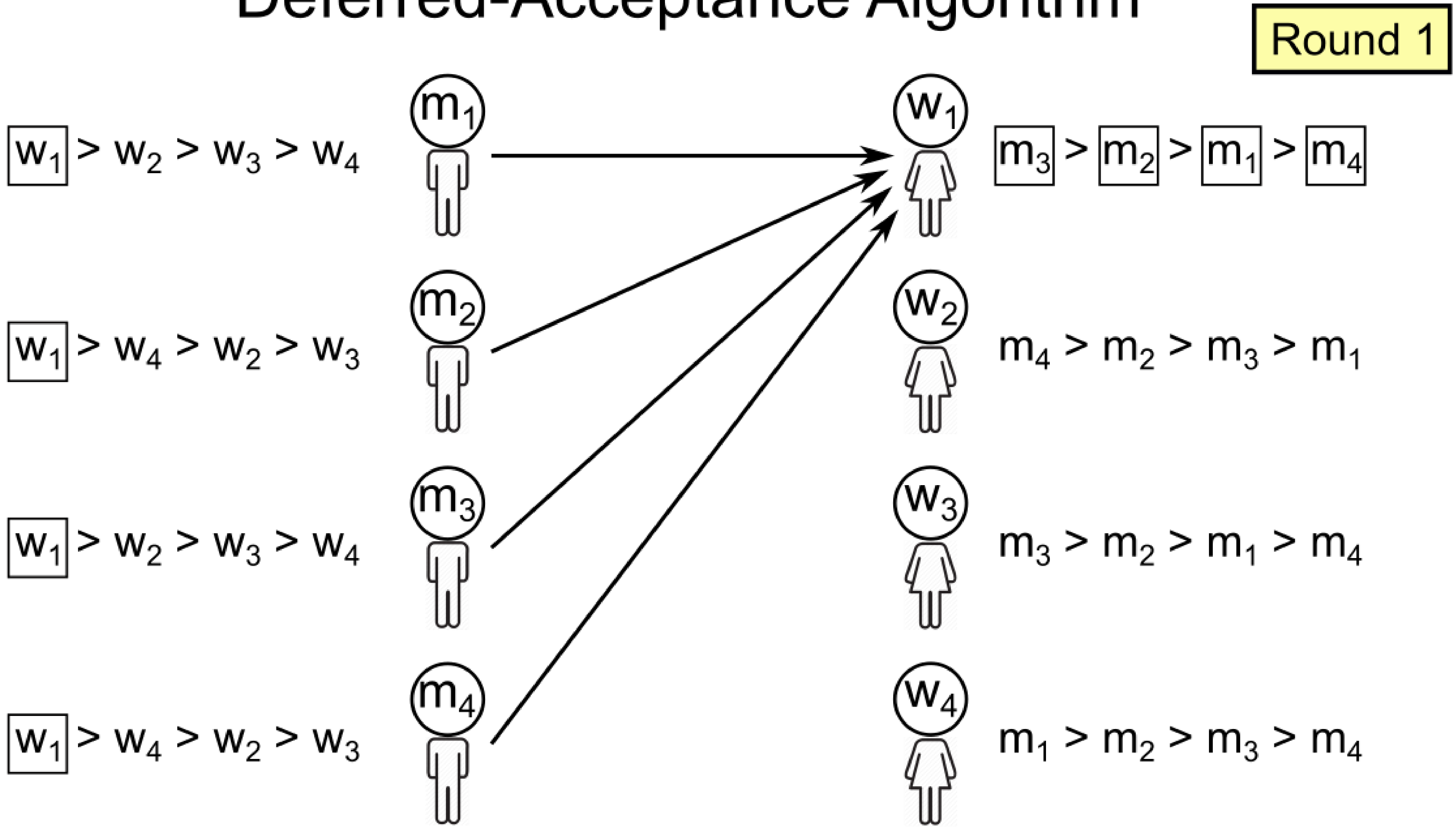

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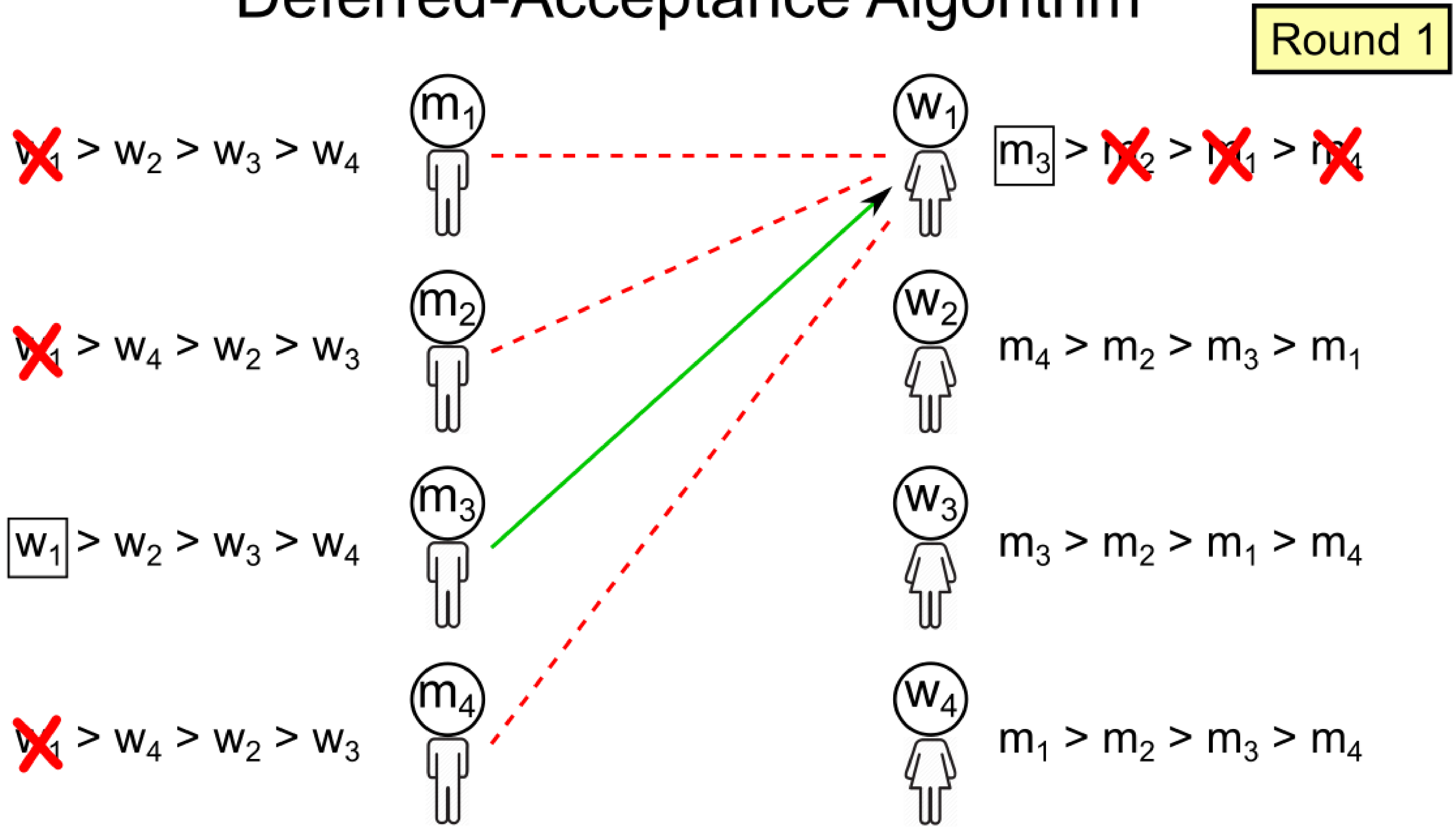
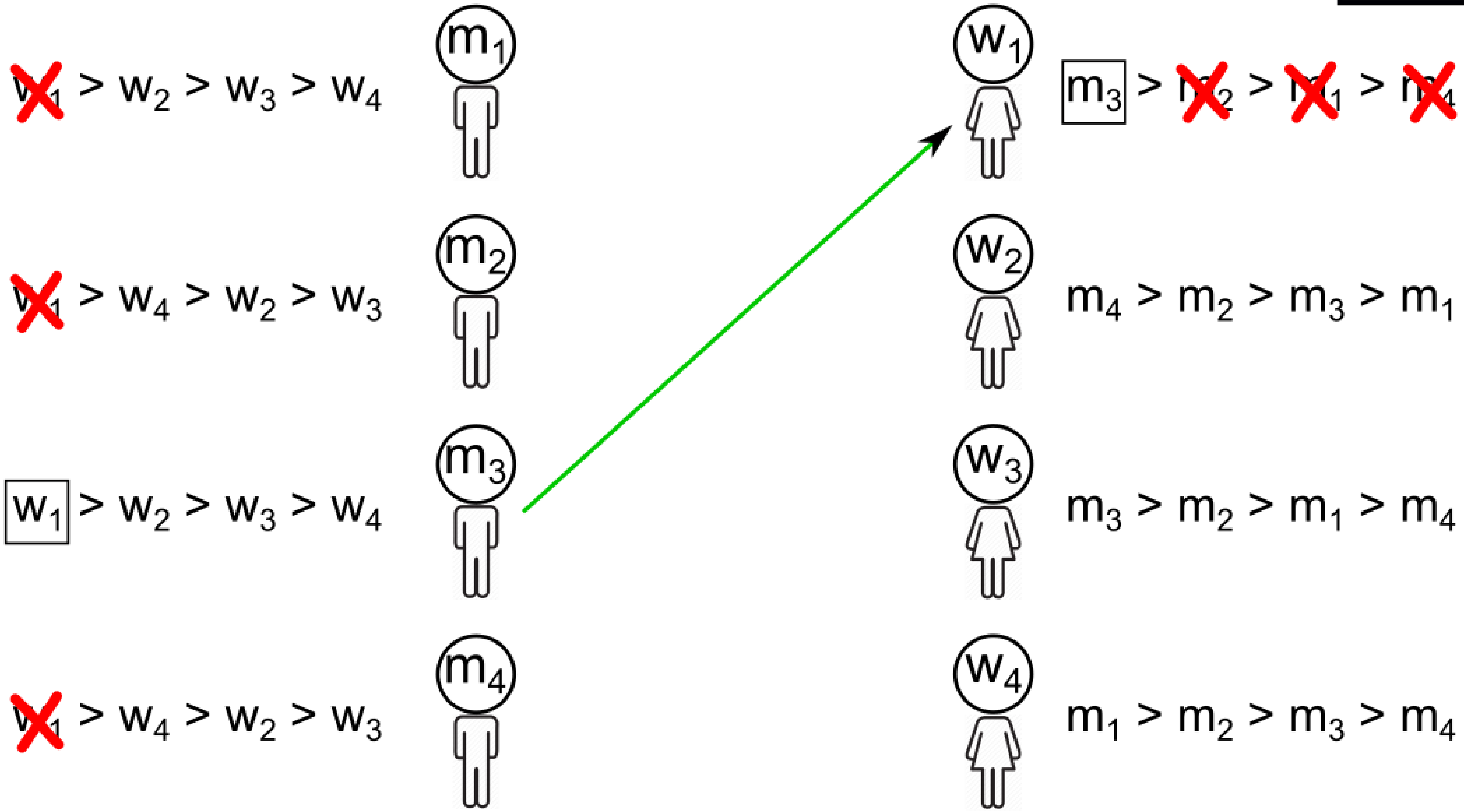


Deferred-Acceptance Algorithm

Round 1



Deferred-Acceptance Algorithm

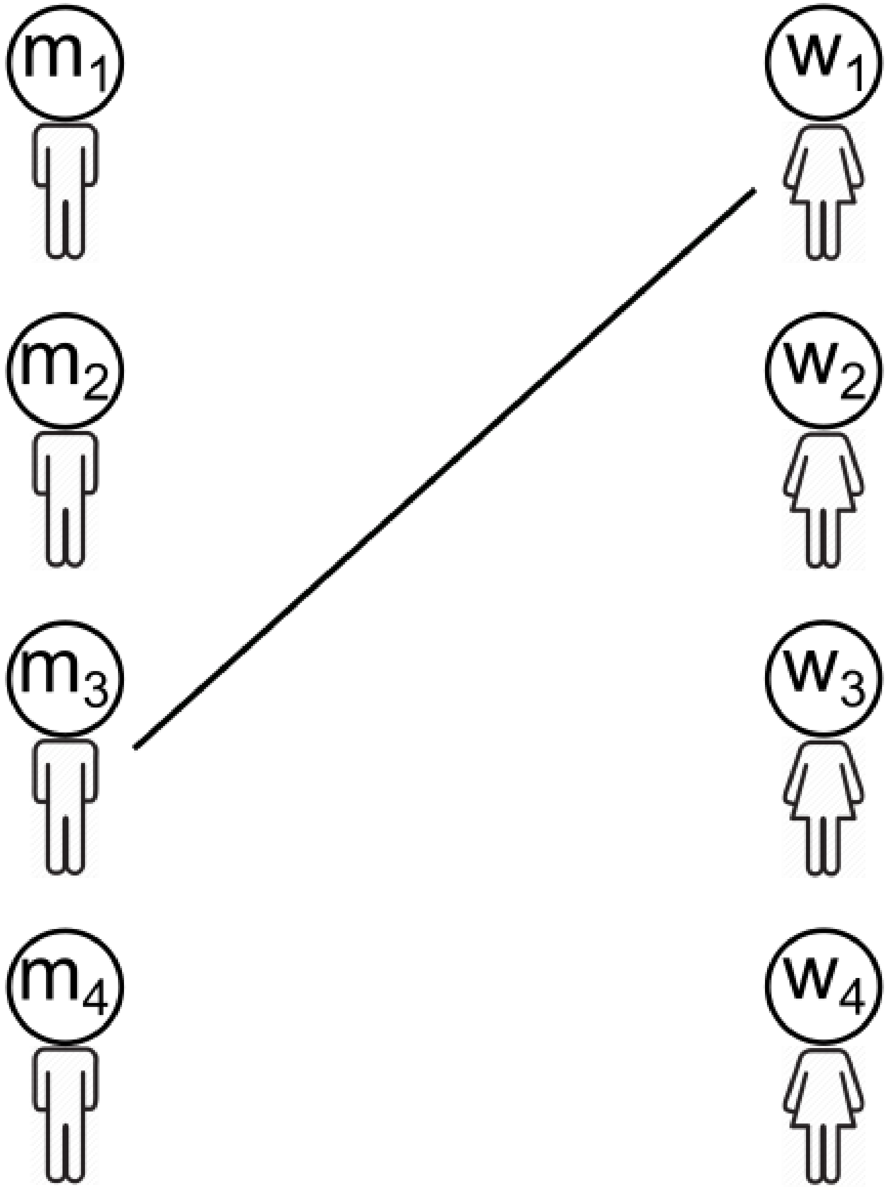
Round 1

~~w_1~~ $> w_2 > w_3 > w_4$

~~w_1~~ $> w_4 > w_2 > w_3$

[w_1] $> w_2 > w_3 > w_4$

~~w_1~~ $> w_4 > w_2 > w_3$



[m_3] $>$ ~~m_2~~ $>$ ~~m_1~~ $>$ ~~m_4~~

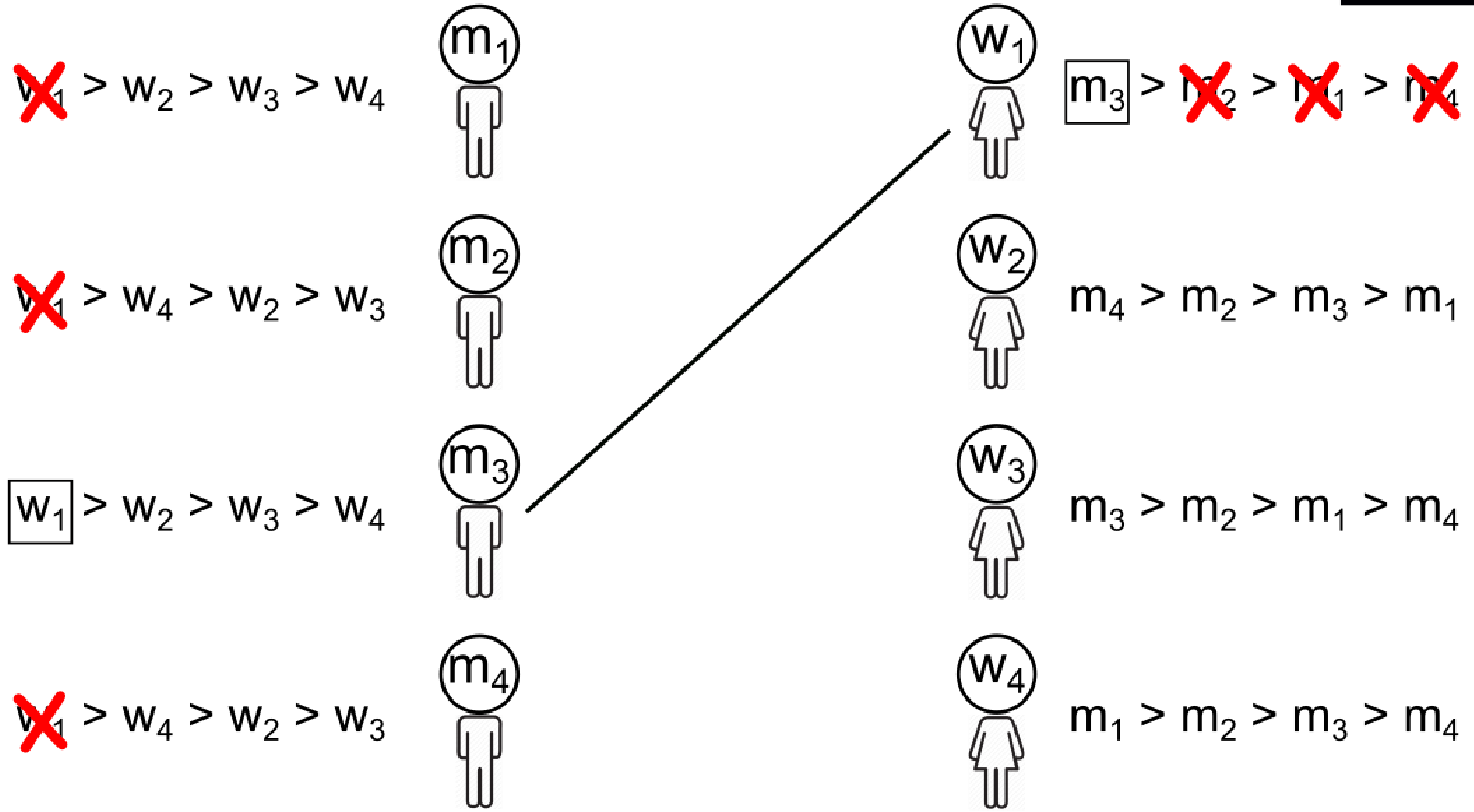
$m_4 > m_2 > m_3 > m_1$

$m_3 > m_2 > m_1 > m_4$

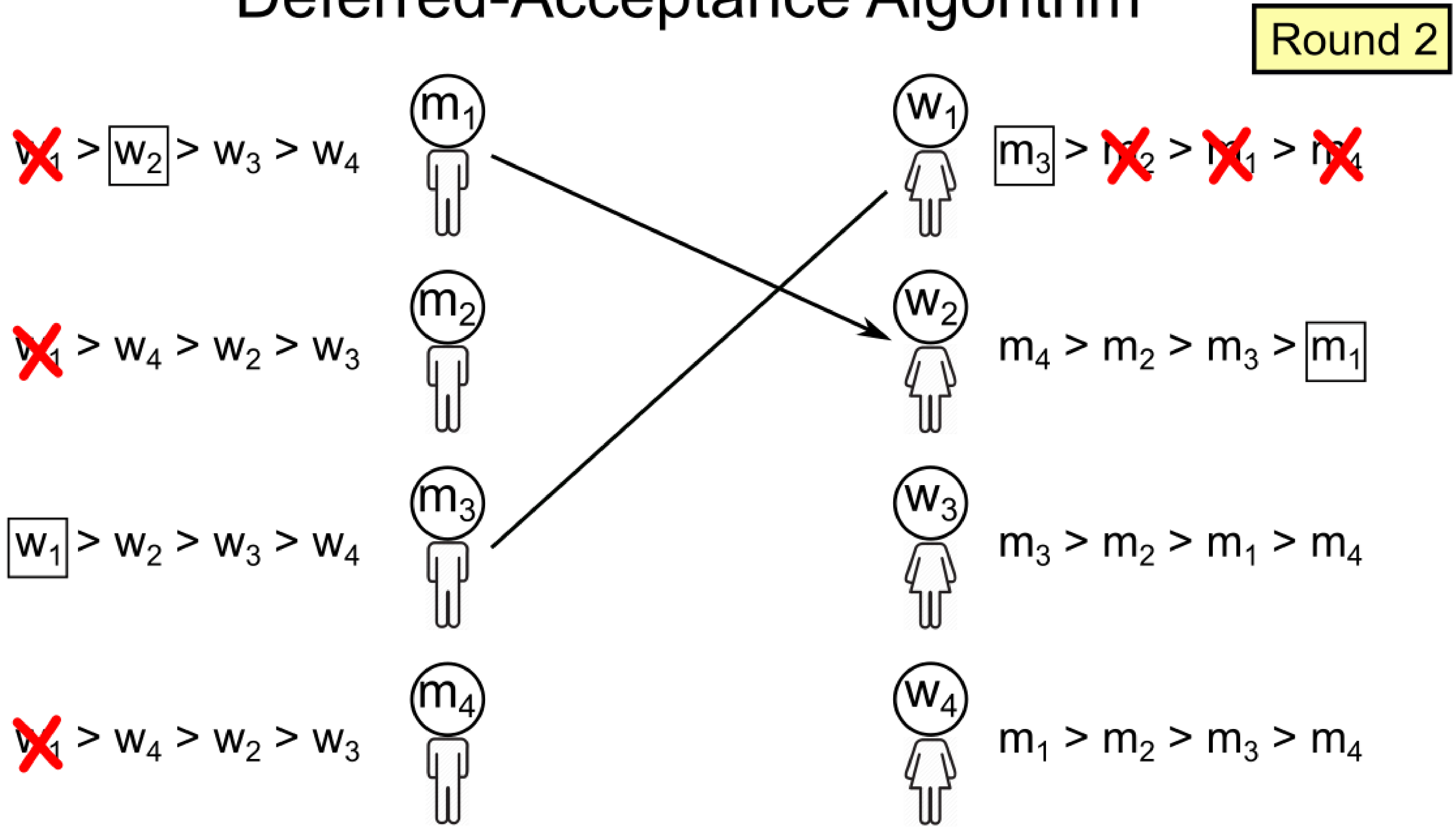
$m_1 > m_2 > m_3 > m_4$

Deferred-Acceptance Algorithm

Round 2



Deferred-Acceptance Algorithm



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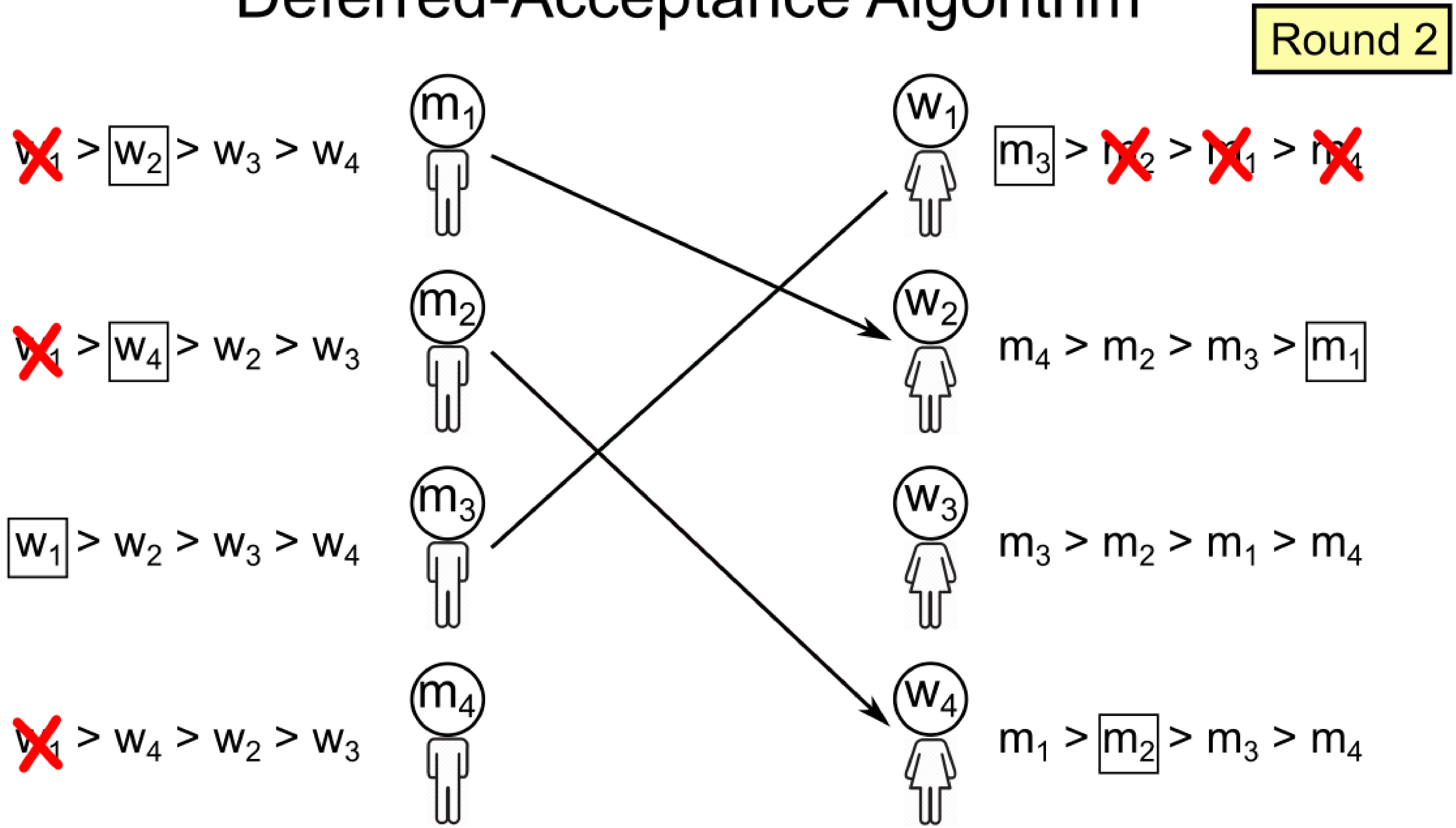

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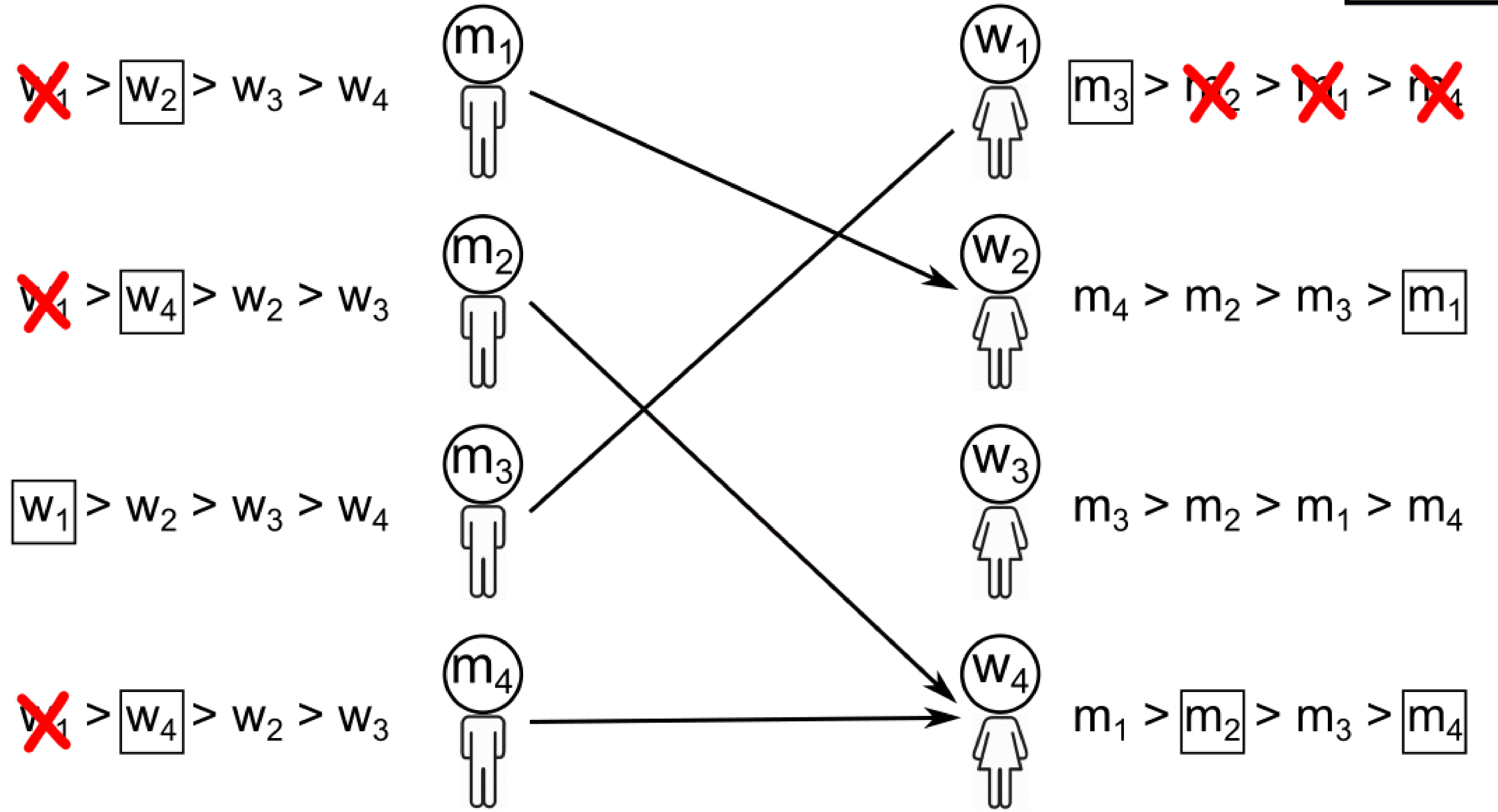
Round 2



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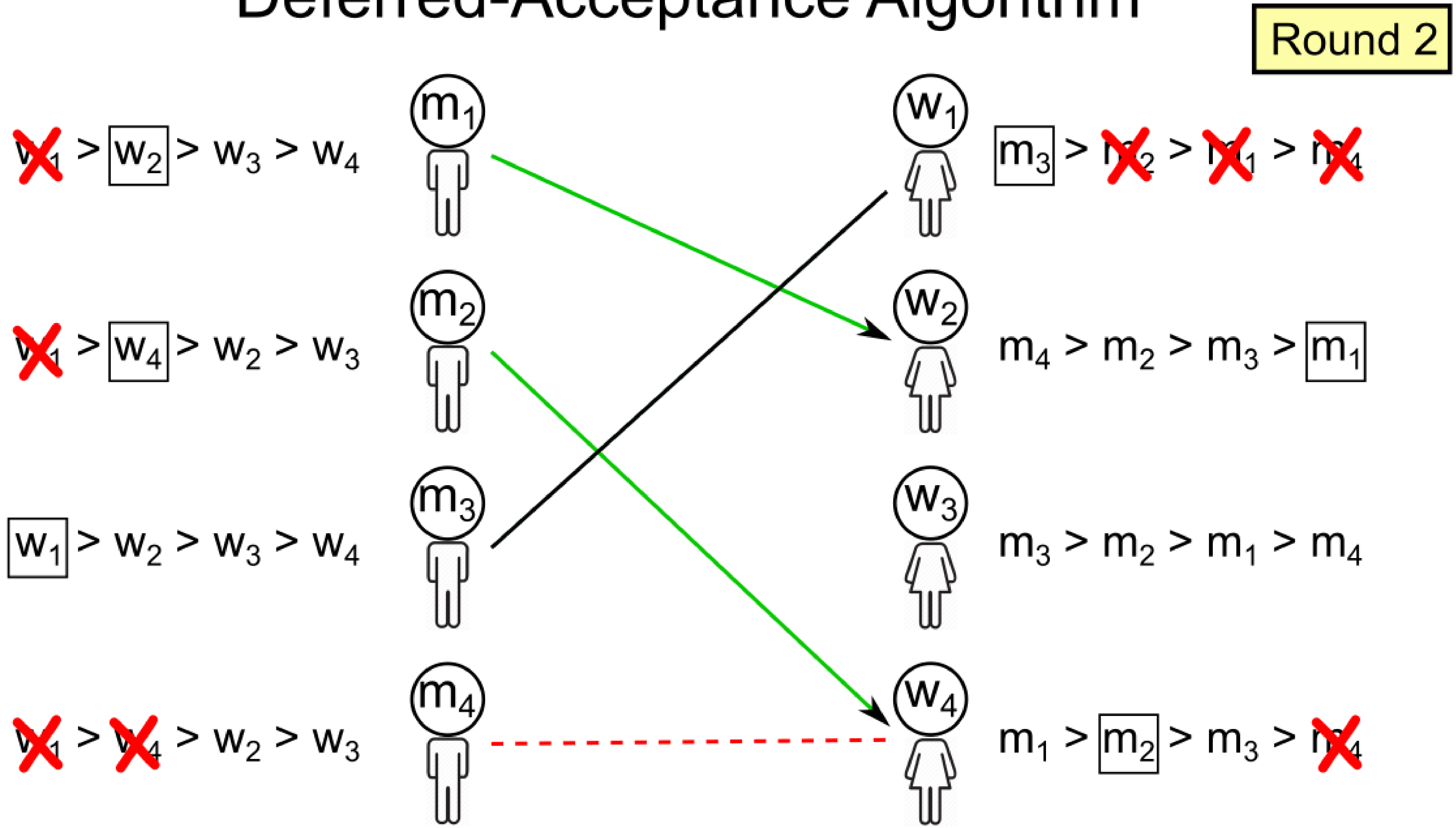

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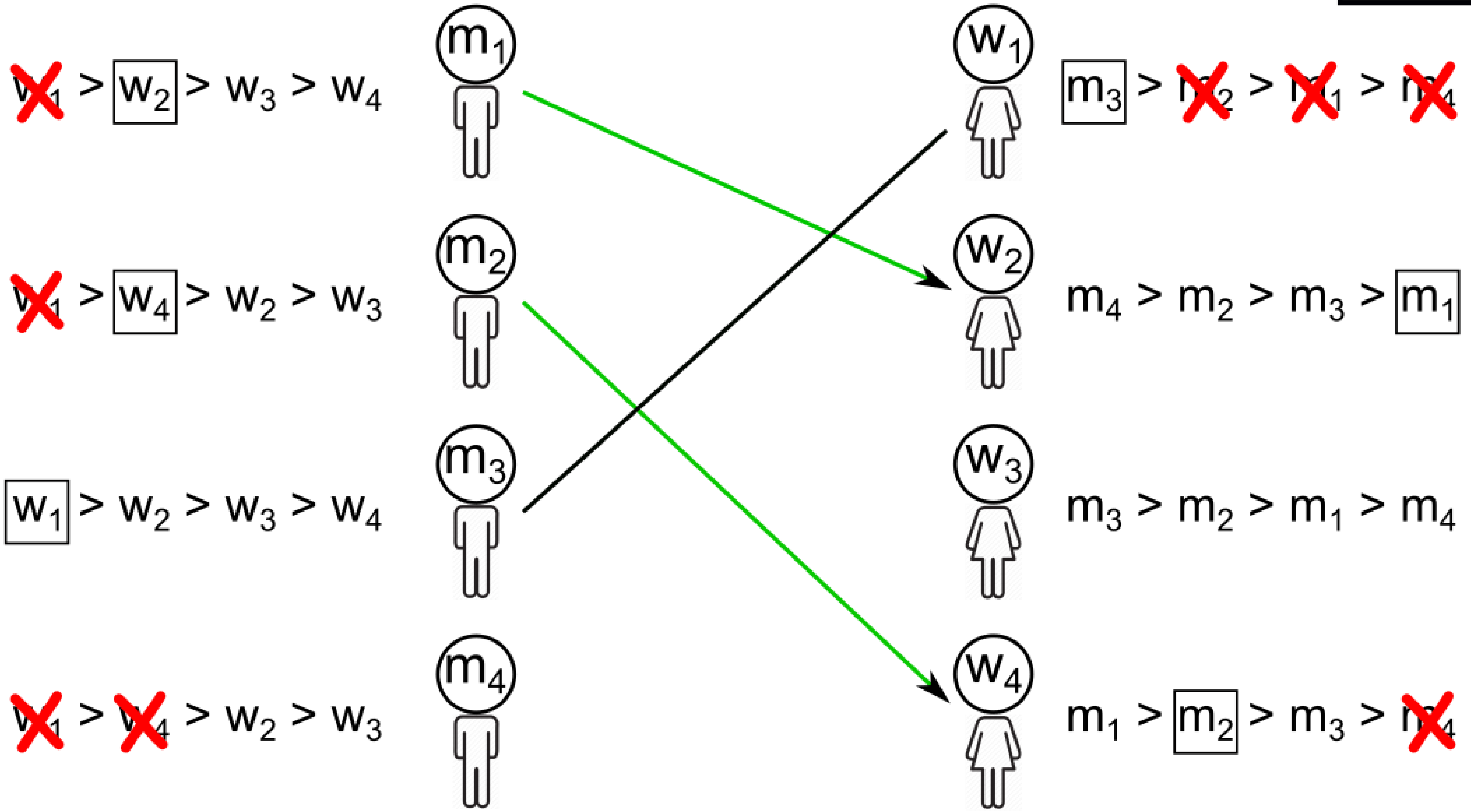

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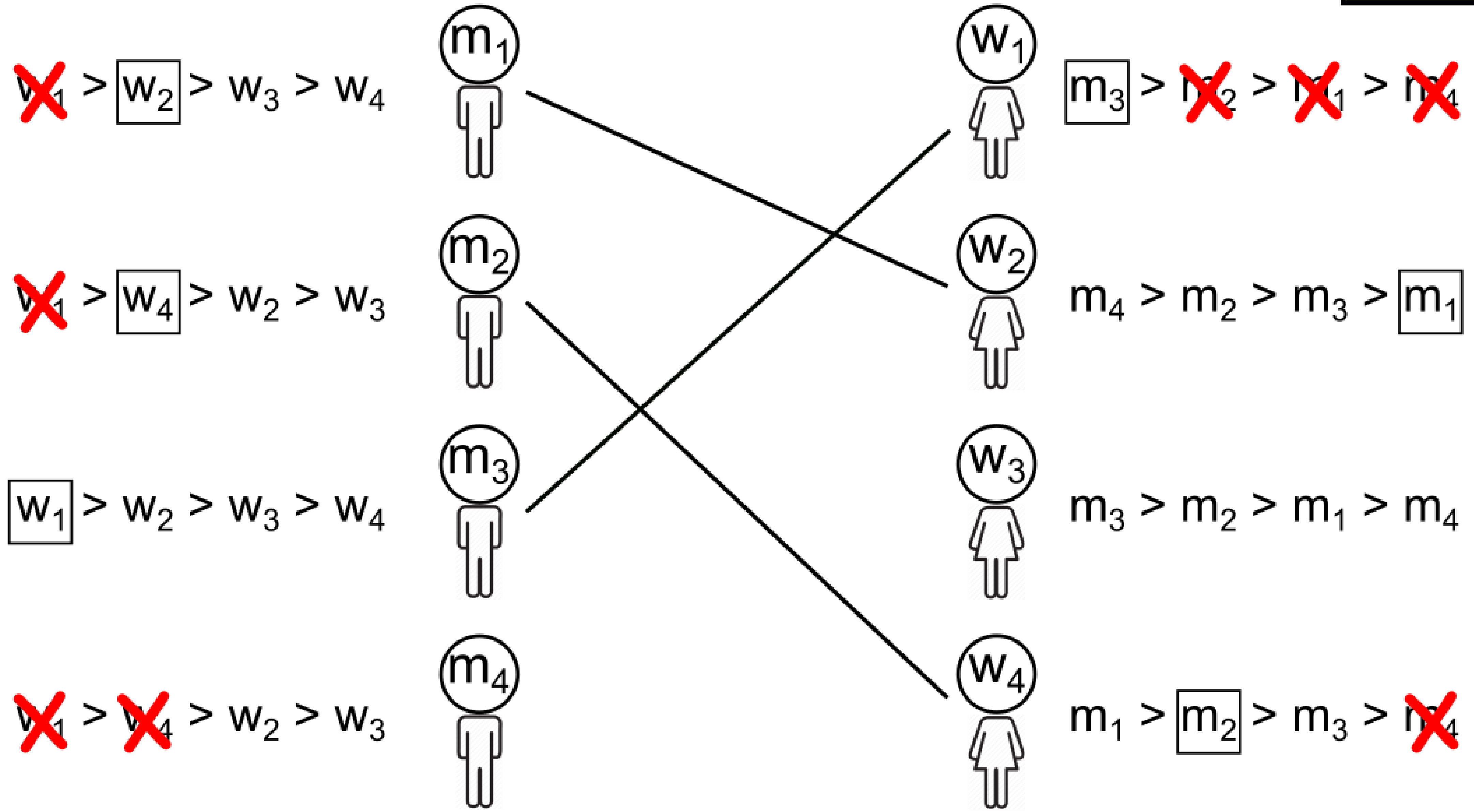

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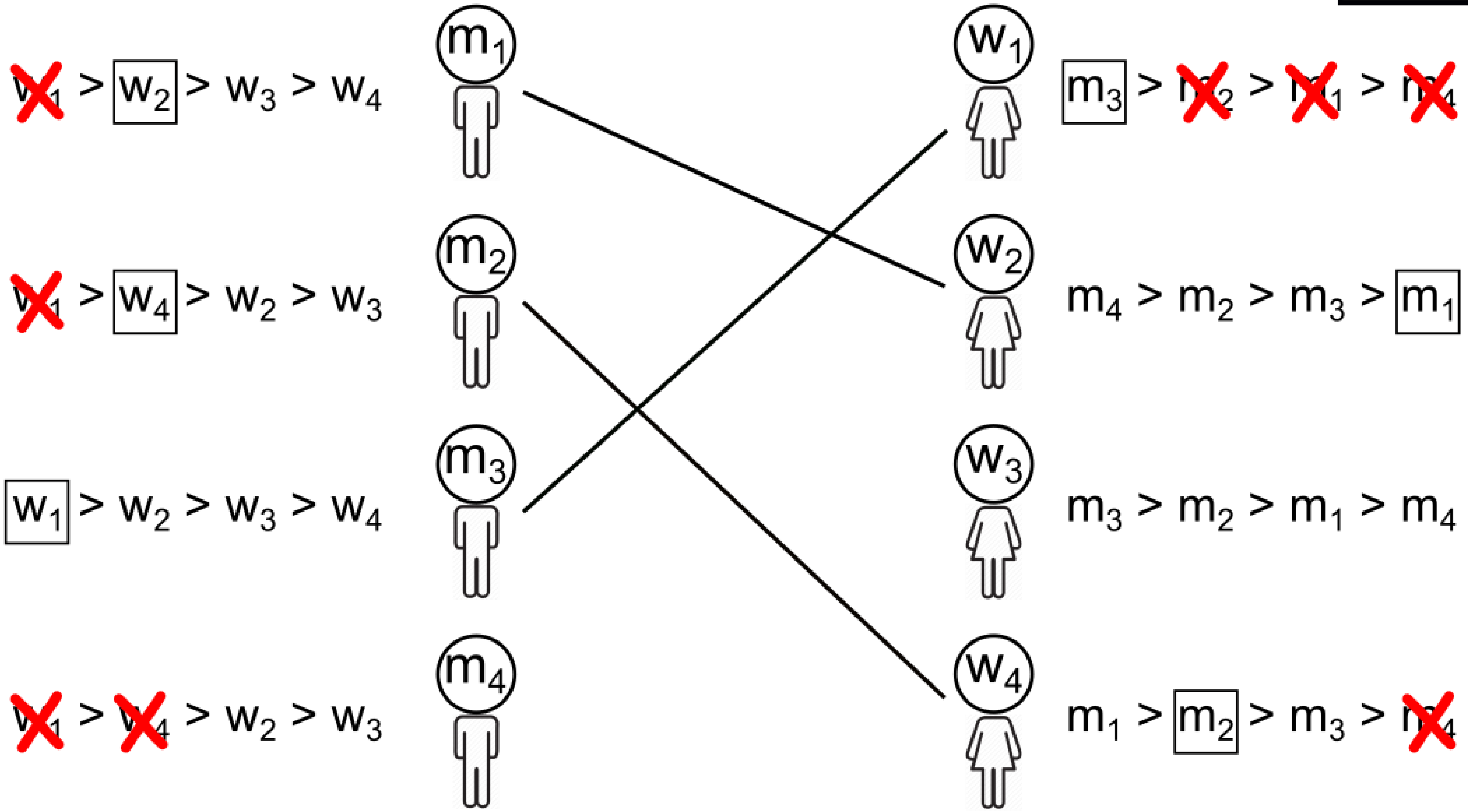
Round 3



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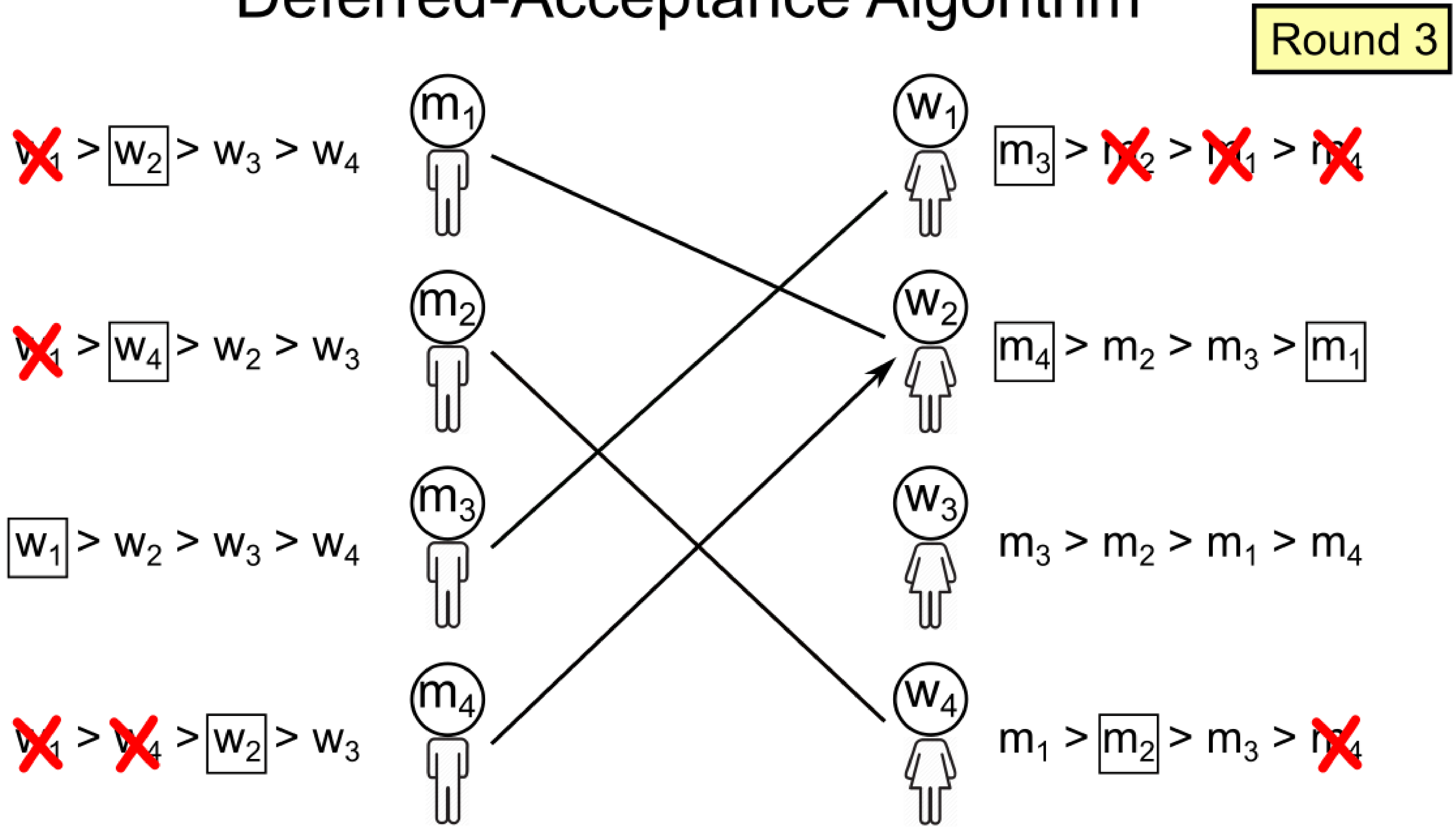

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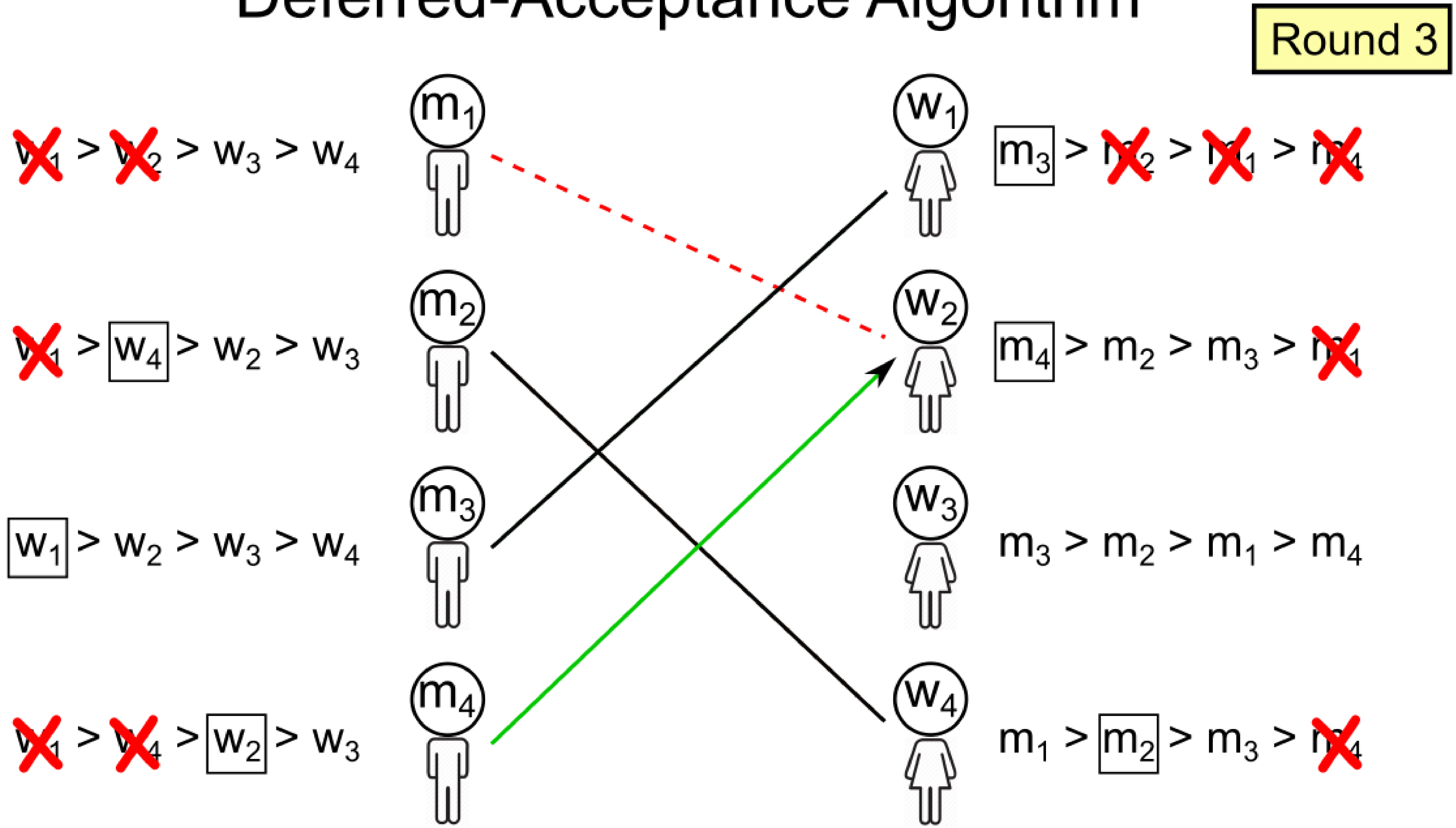

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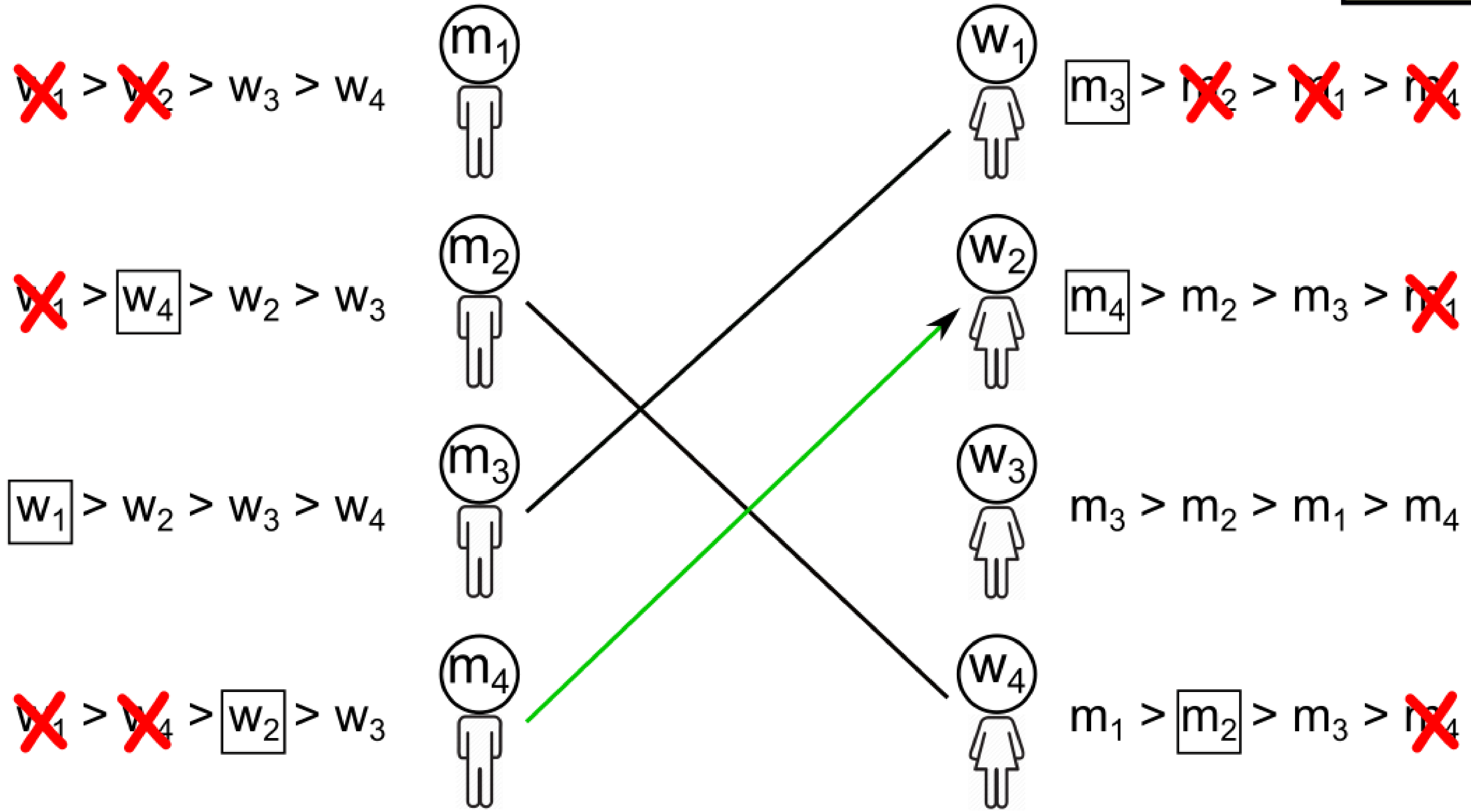
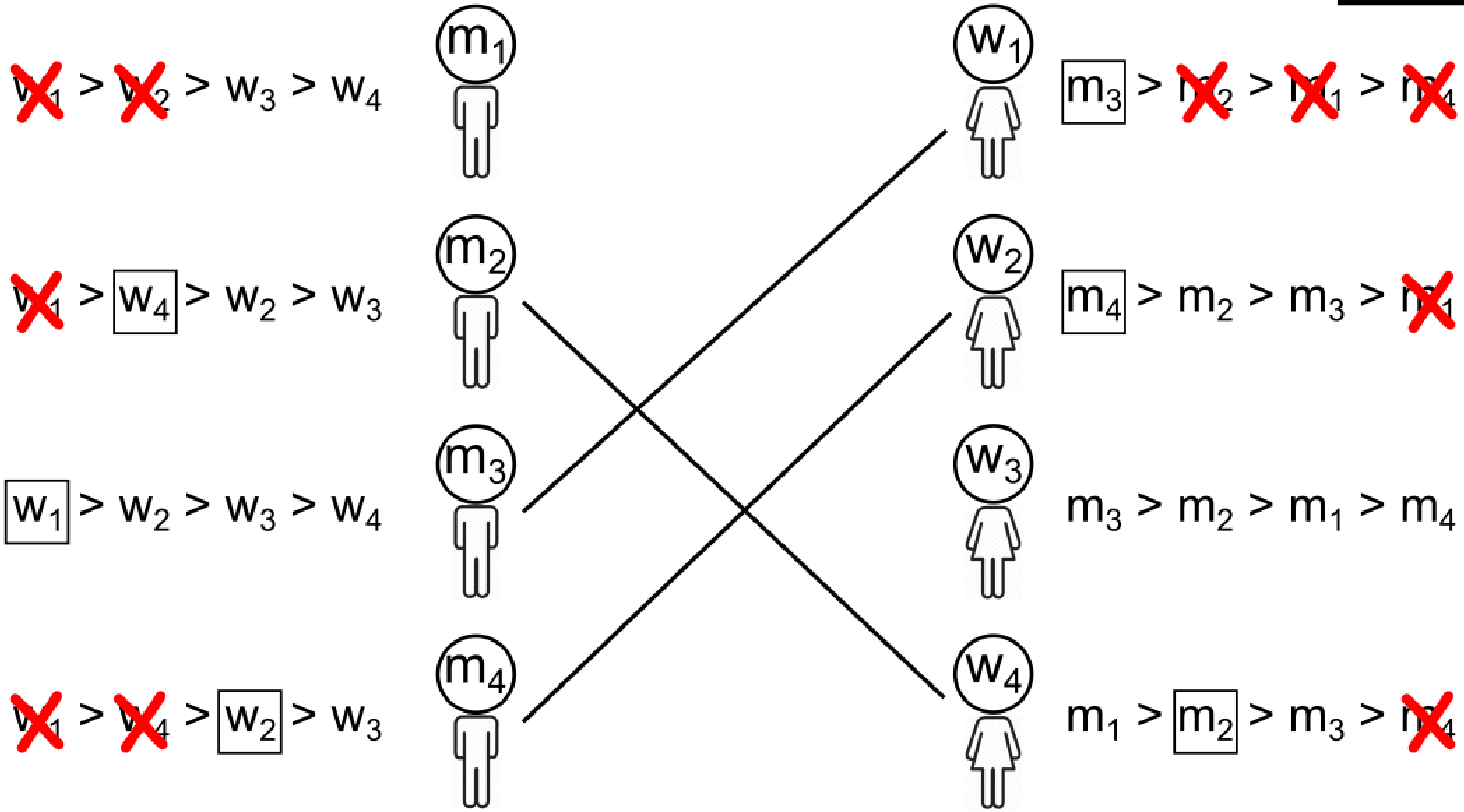


Deferred-Acceptance Algorithm

Round 3



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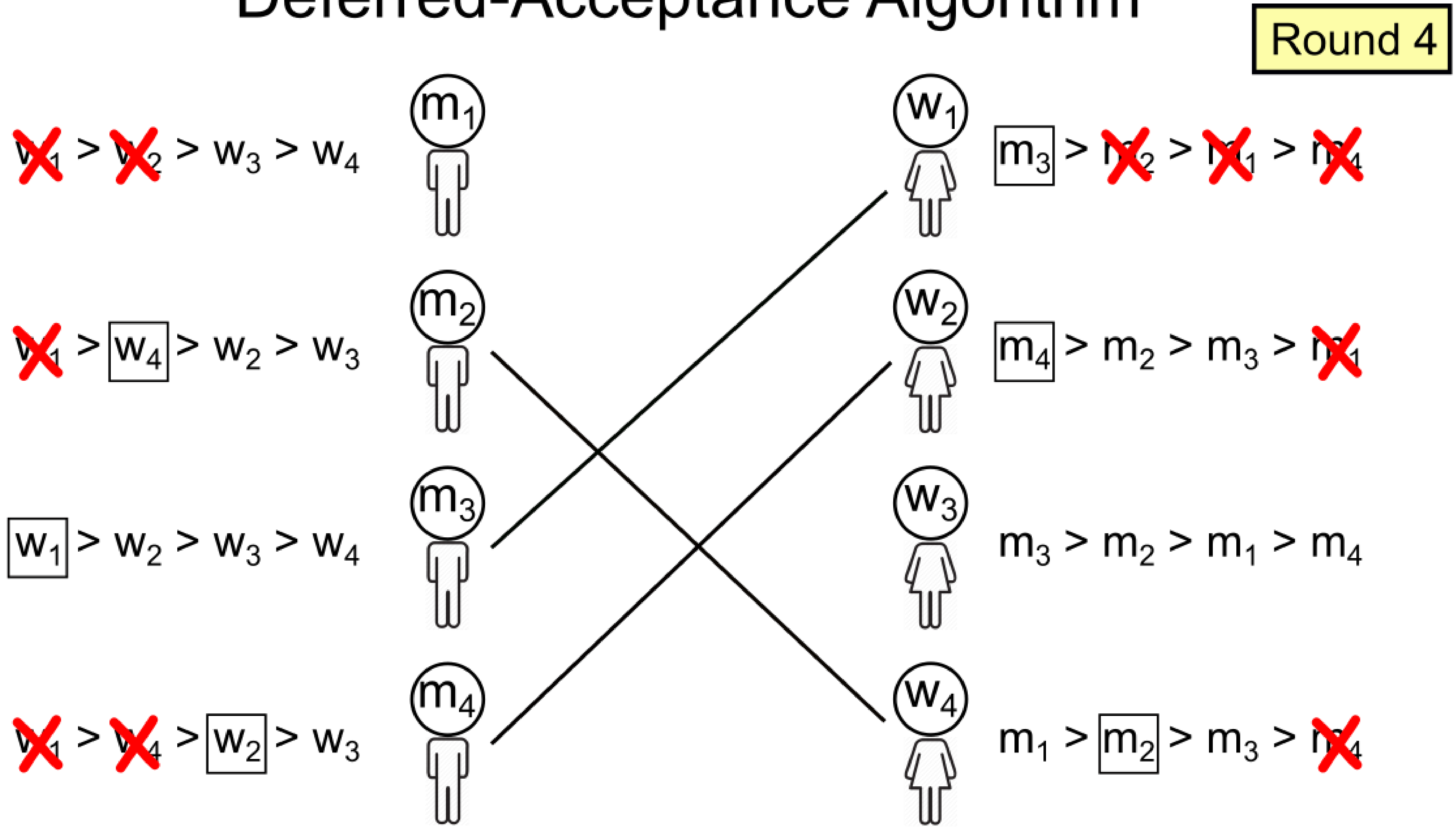

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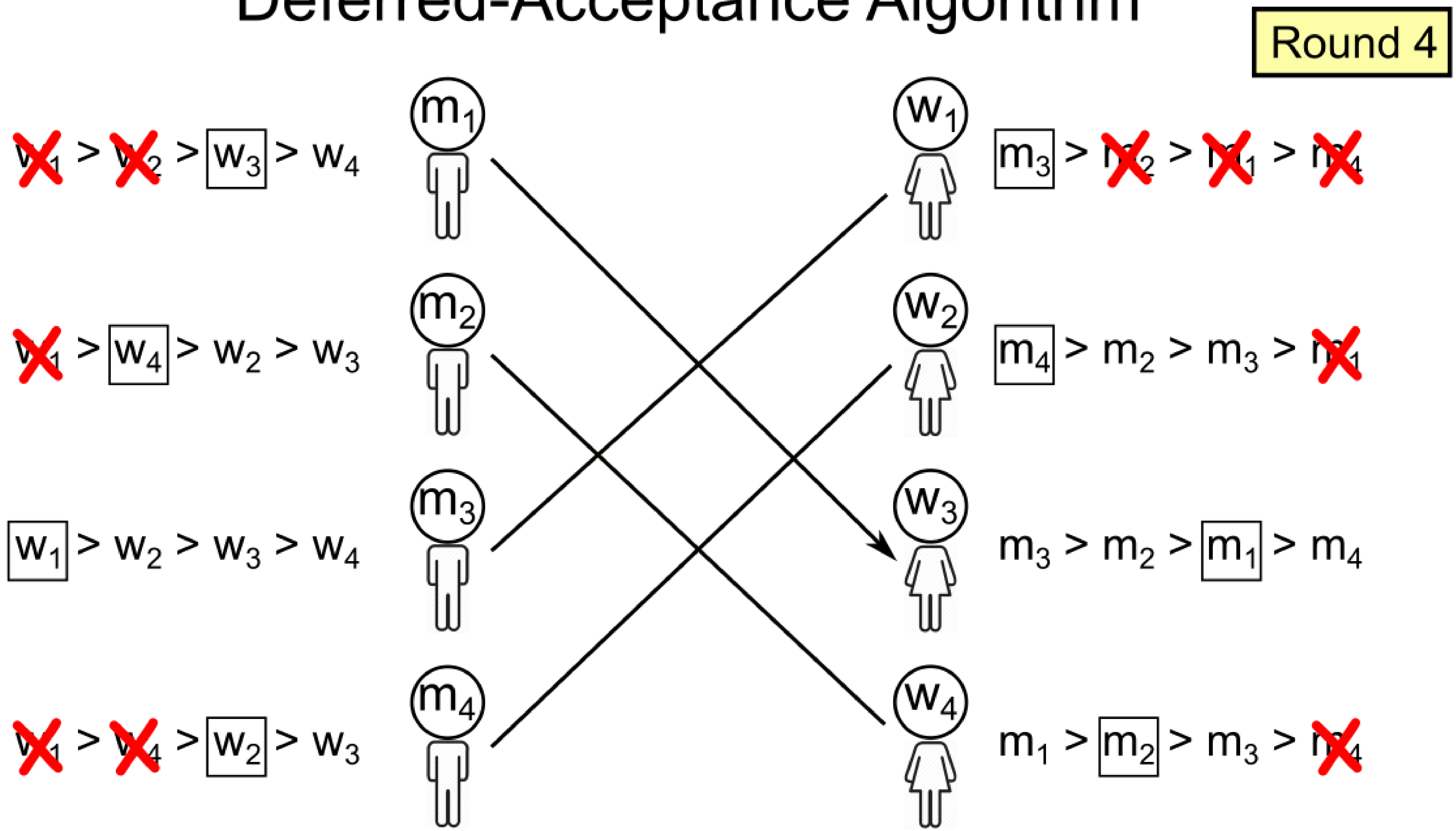
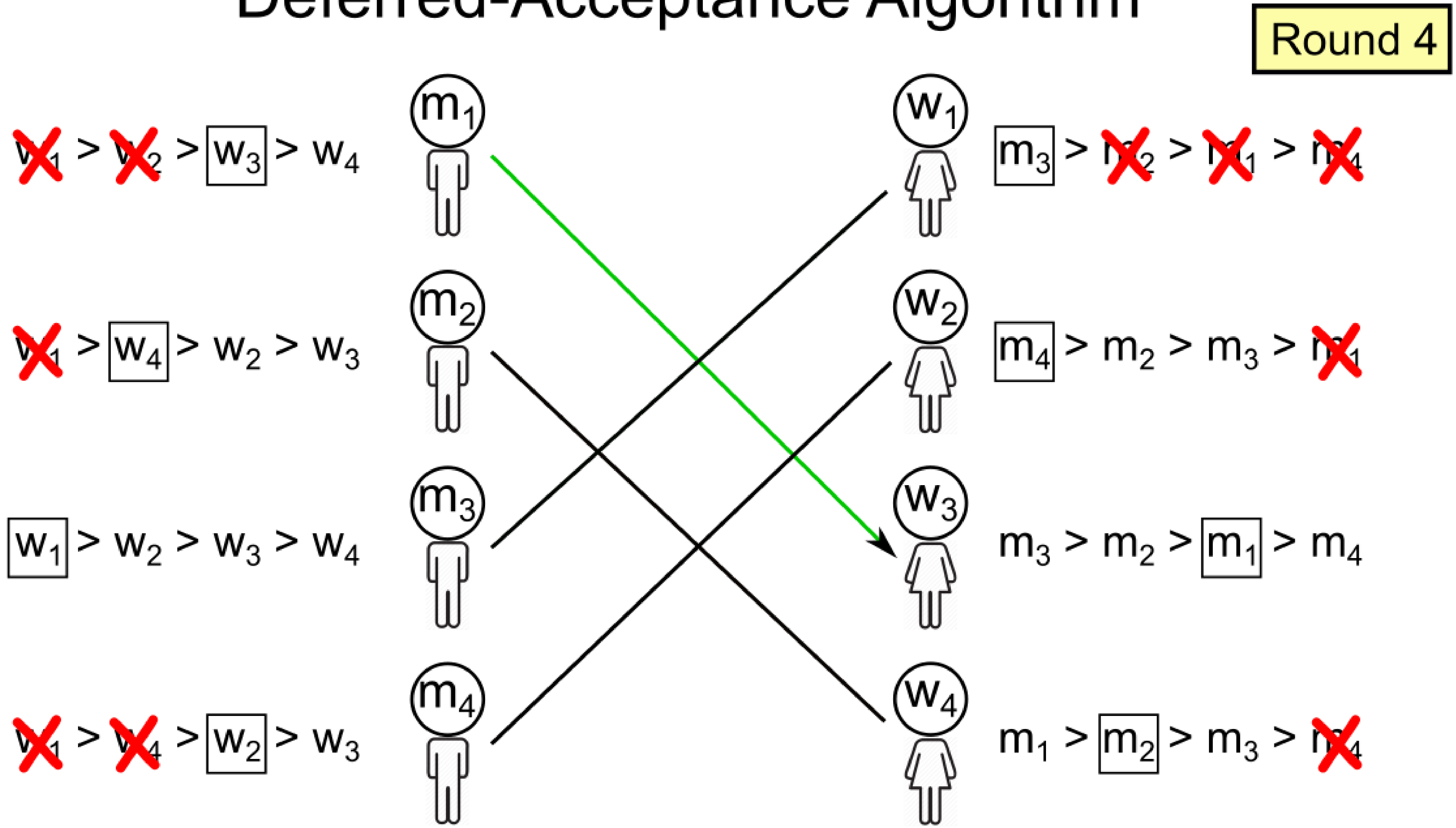
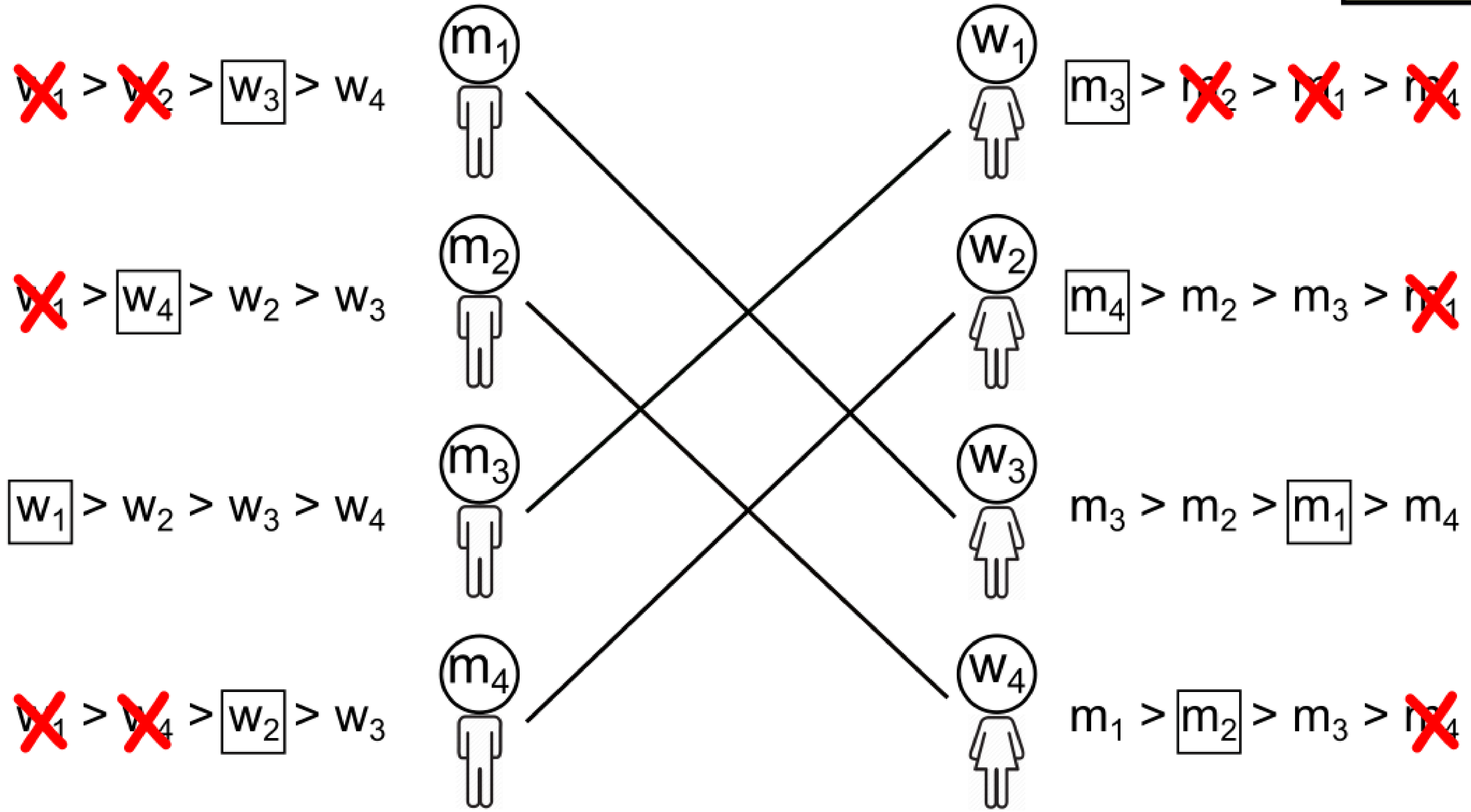


Deferred-Acceptance Algorithm

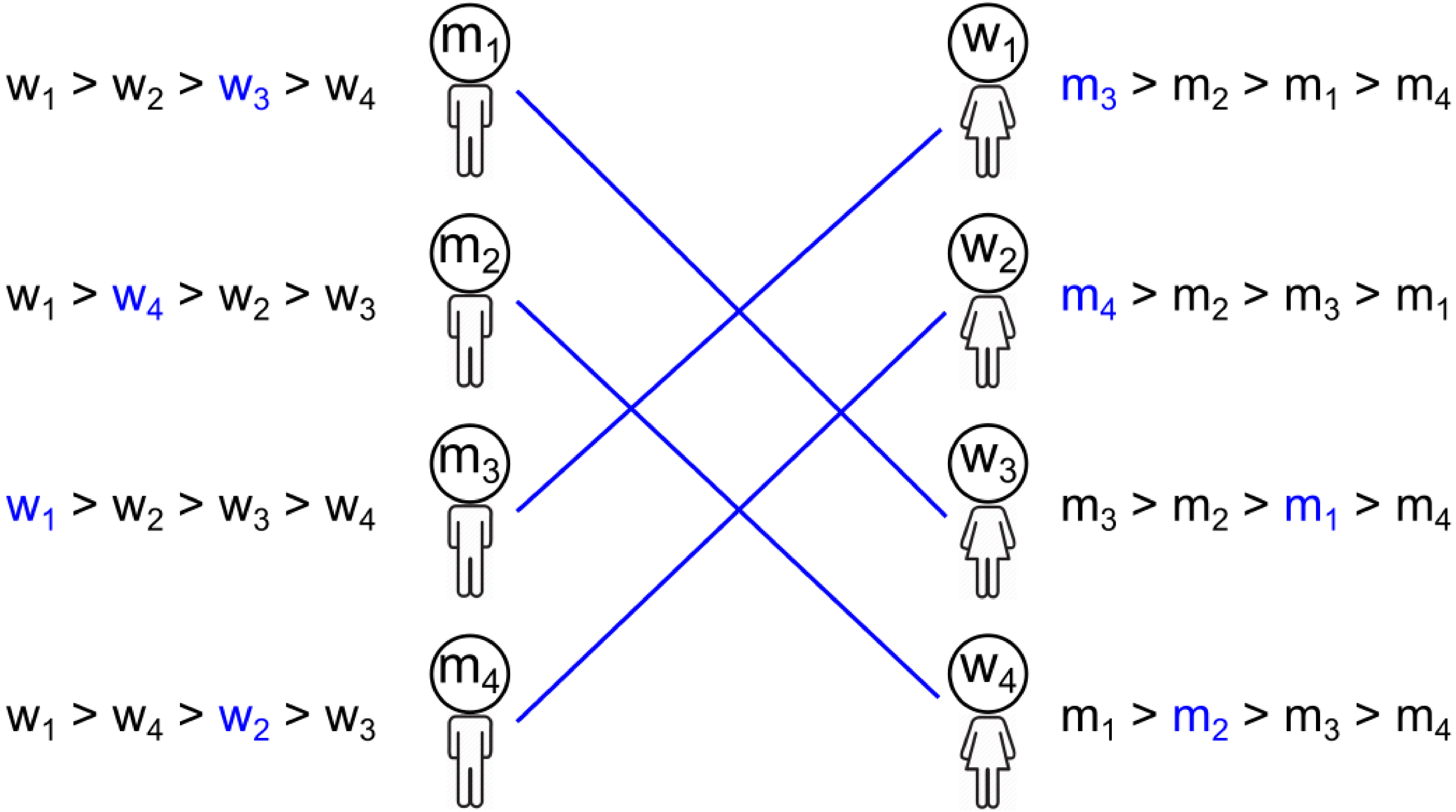


Deferred-Acceptance Algorithm

Round 4



Deferred-Acceptance Algorithm



1

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1

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Deferred-acceptance algorithm terminates in polynomial time.

2

Does the deferred-acceptance algorithm always find a perfect matching?

2

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Man m must have proposed to (and been rejected by) woman w, meaning w got a better-than-m proposal in some round.

Once tentatively matched, a woman never becomes unmatched.

3

Does the deferred-acceptance algorithm always find a stable matching?

3

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3

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Then, w must have received a better-than-m proposal in some round.

Women only "trade up" during the DA algorithm.


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Trends in Computational Social Choice

CHAPTER 18

Applications of Matching Models under Preferences

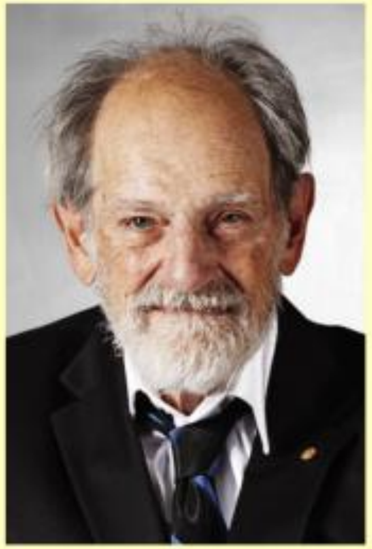
Péter Biró

18.1 Introduction

Matching problems under preferences have been studied widely in mathematics, computer science and economics, starting with the seminal paper by Gale and Shapley (1962). A comprehensive survey on this topic was published also in Chapter 14 of the Handbook of Computational Social Choice (Klaus et al., 2016), and for the interested reader we recommend consulting the following four comprehensive books on the computational (Gusfield and Irving, 1989; Manlove, 2013) and game-theoretical, market design aspects (Roth and Sotomayor, 1990; Roth, 2015) of this topic. In this chapter our goal is to give a general overview of the related applications.

Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

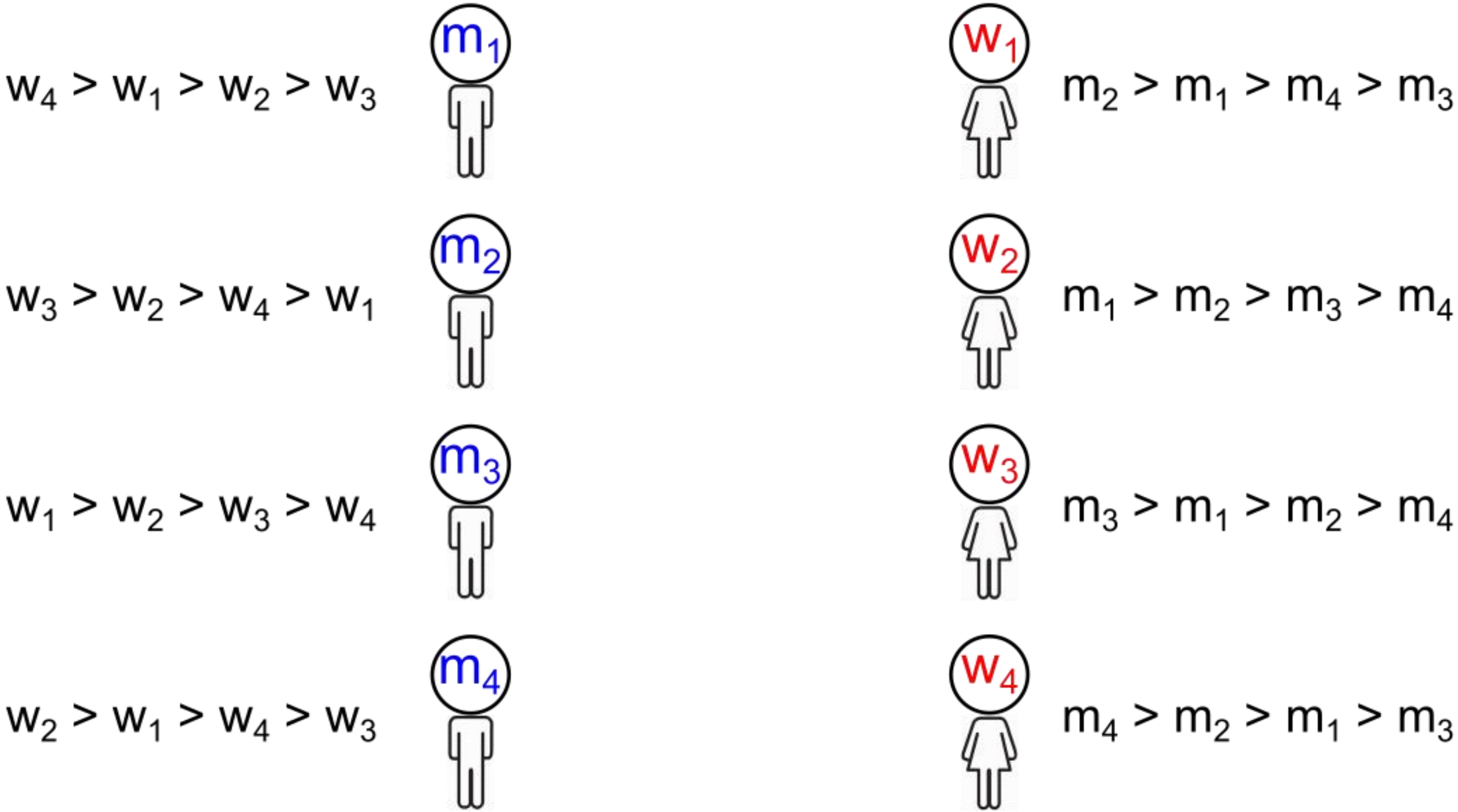
Alvin E. Roth

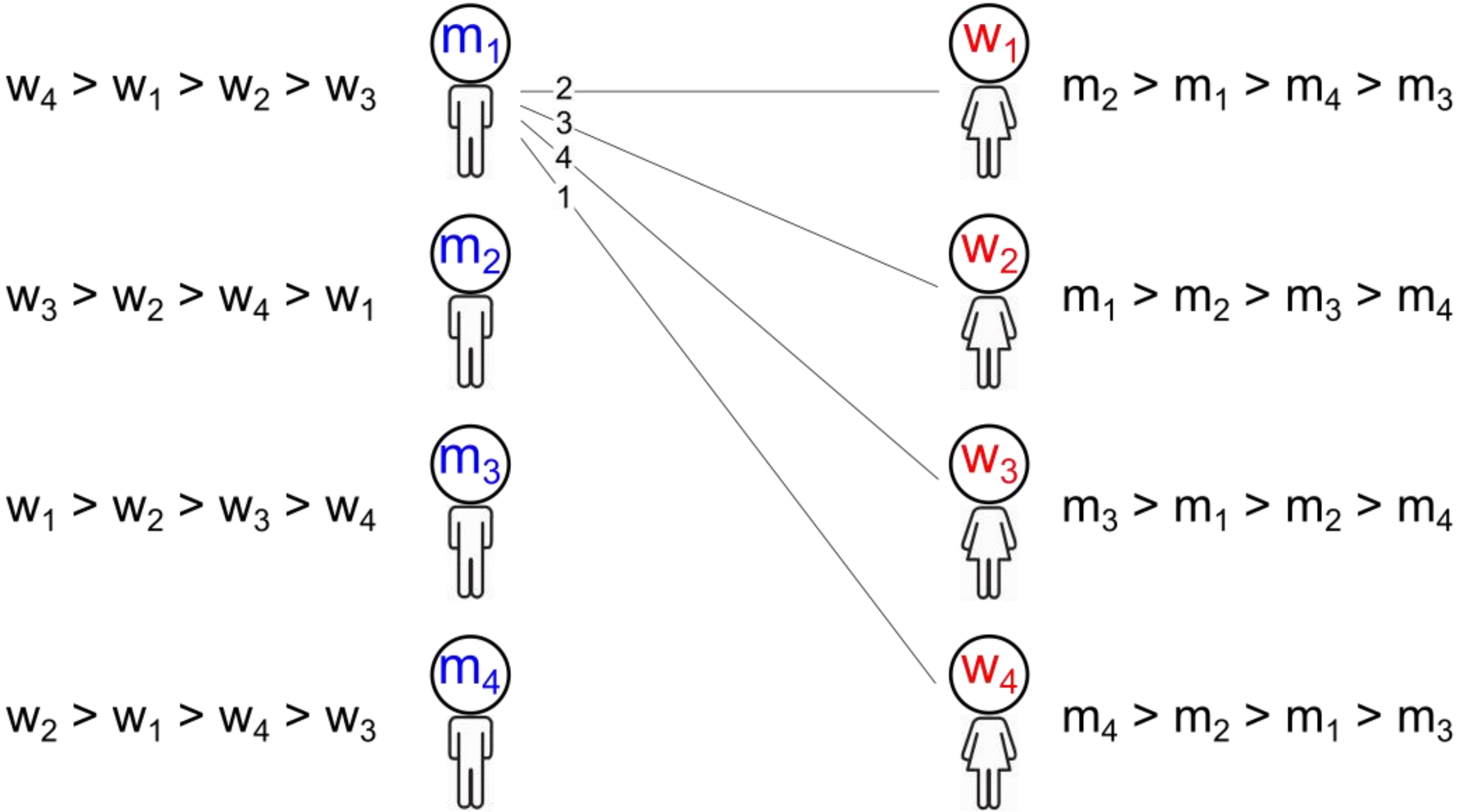


Lloyd S. Shapley

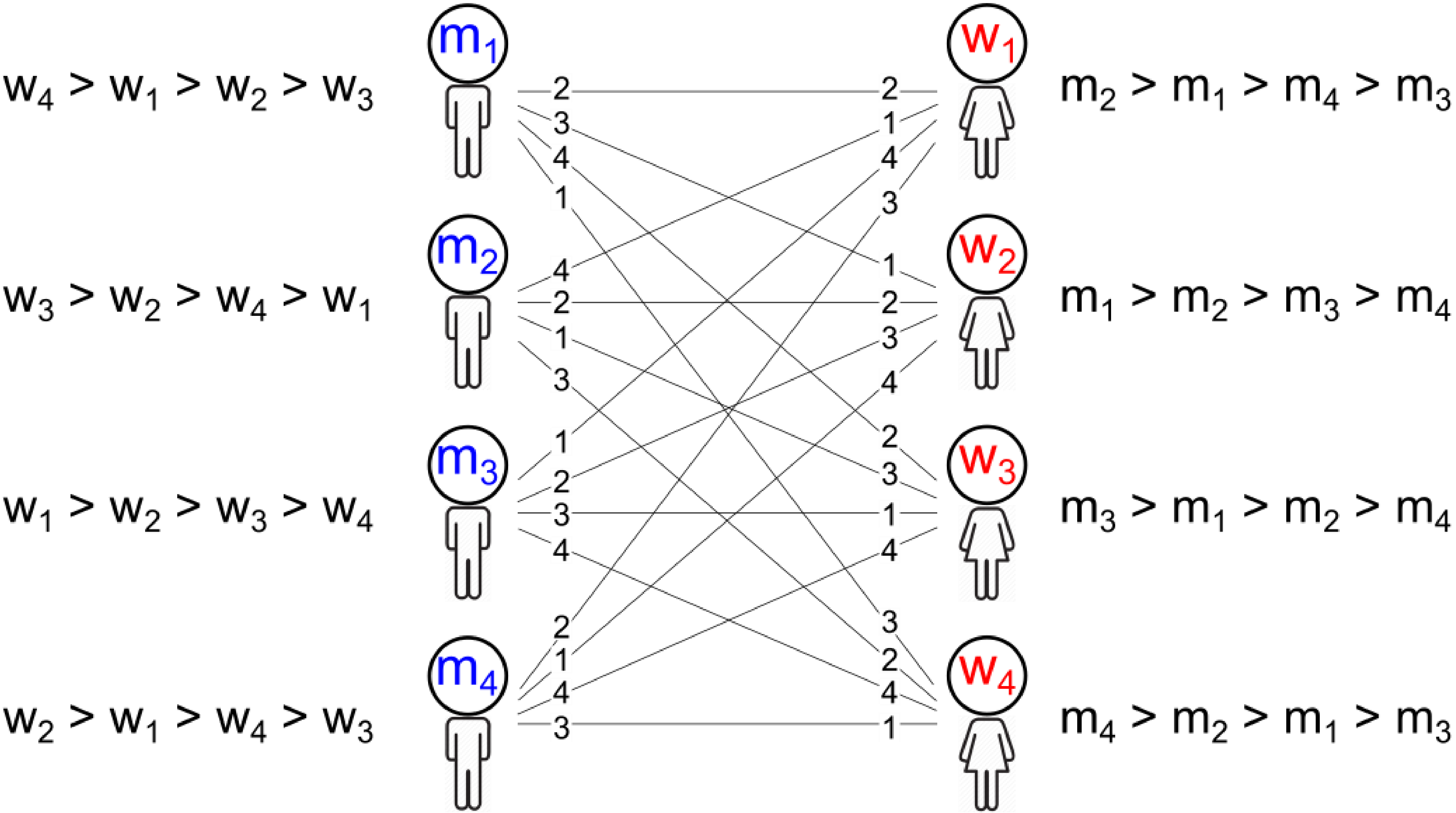
"for the theory of stable allocations and the practice of market design."

Structure of the Set of Stable Matchings


$w_4 > w_1 > w_2 > w_3$
m_1
w_1
$m_2 > m_1 > m_4 > m_3$
$w_3 > w_2 > w_4 > w_1$
m_2
w_2
$m_1 > m_2 > m_3 > m_4$
$w_1 > w_2 > w_3 > w_4$
m_3
w_3
$m_3 > m_1 > m_2 > m_4$
$w_2 > w_1 > w_4 > w_3$
m_4
w_4
$m_4 > m_2 > m_1 > m_3$



m1
m2
m3
m4
w1
w2
w3
w4
w4 > w1 > w2 > w3
w3 > w2 > w4 > w1
w1 > w2 > w3 > w4
w2 > w1 > w4 > w3
m2 > m1 > m4 > m3
m1 > m2 > m3 > m4
m3 > m1 > m2 > m4
m4 > m2 > m1 > m3
2
3
4
1



w4 > w1 > w2 > w3
m1
w1
m2 > m1 > m4 > m3
w3 > w2 > w4 > w1
m2
w2
m1 > m2 > m3 > m4
w1 > w2 > w3 > w4
m3
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m3 > m1 > m2 > m4
w2 > w1 > w4 > w3
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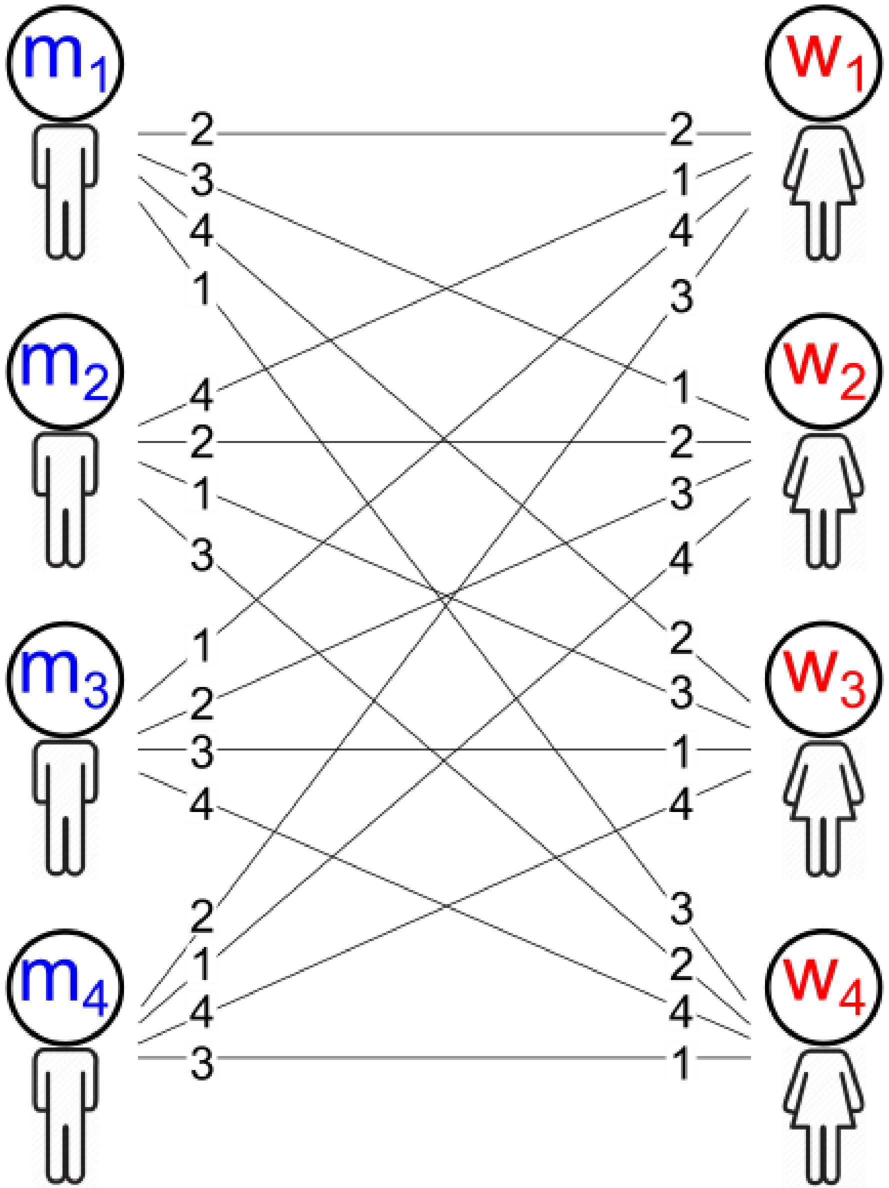
m4 > m2 > m1 > m3



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m3
m4
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w4
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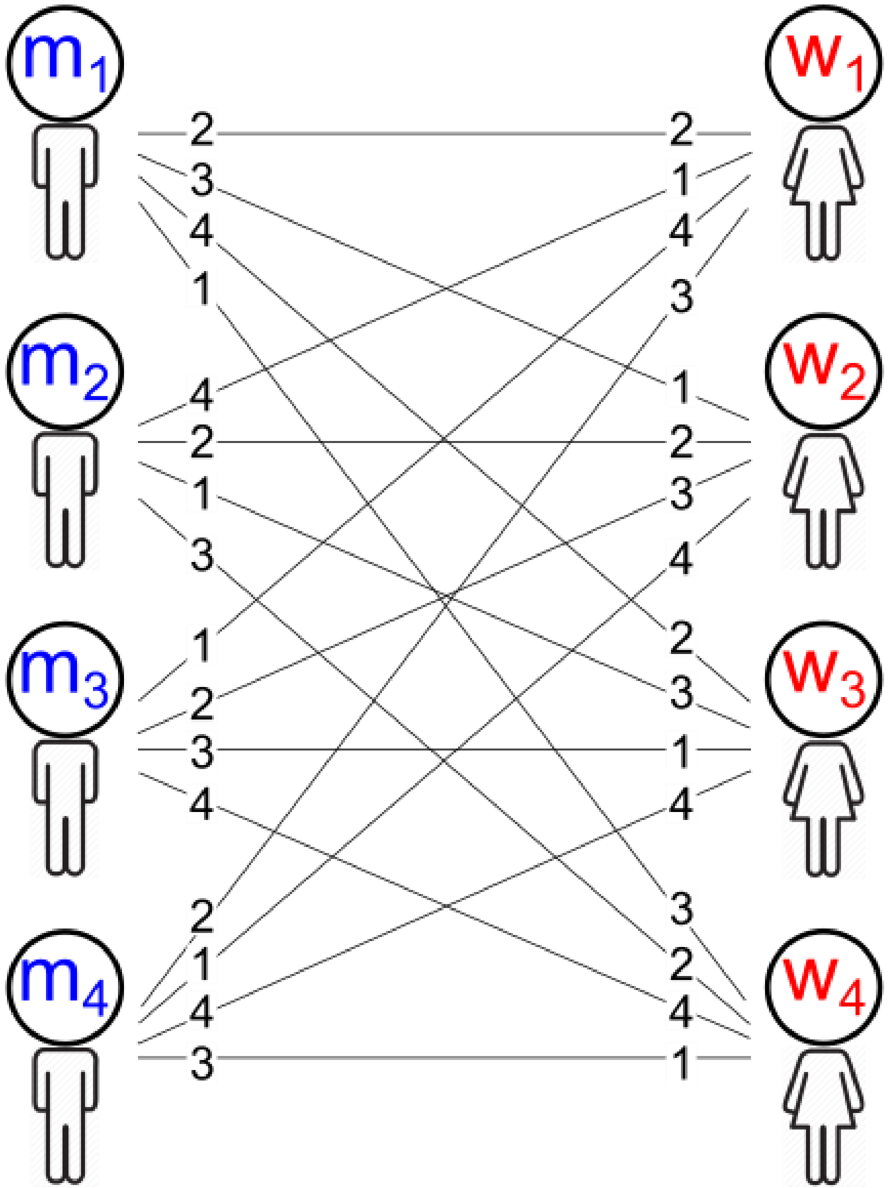
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m3
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m4
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w1
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w4
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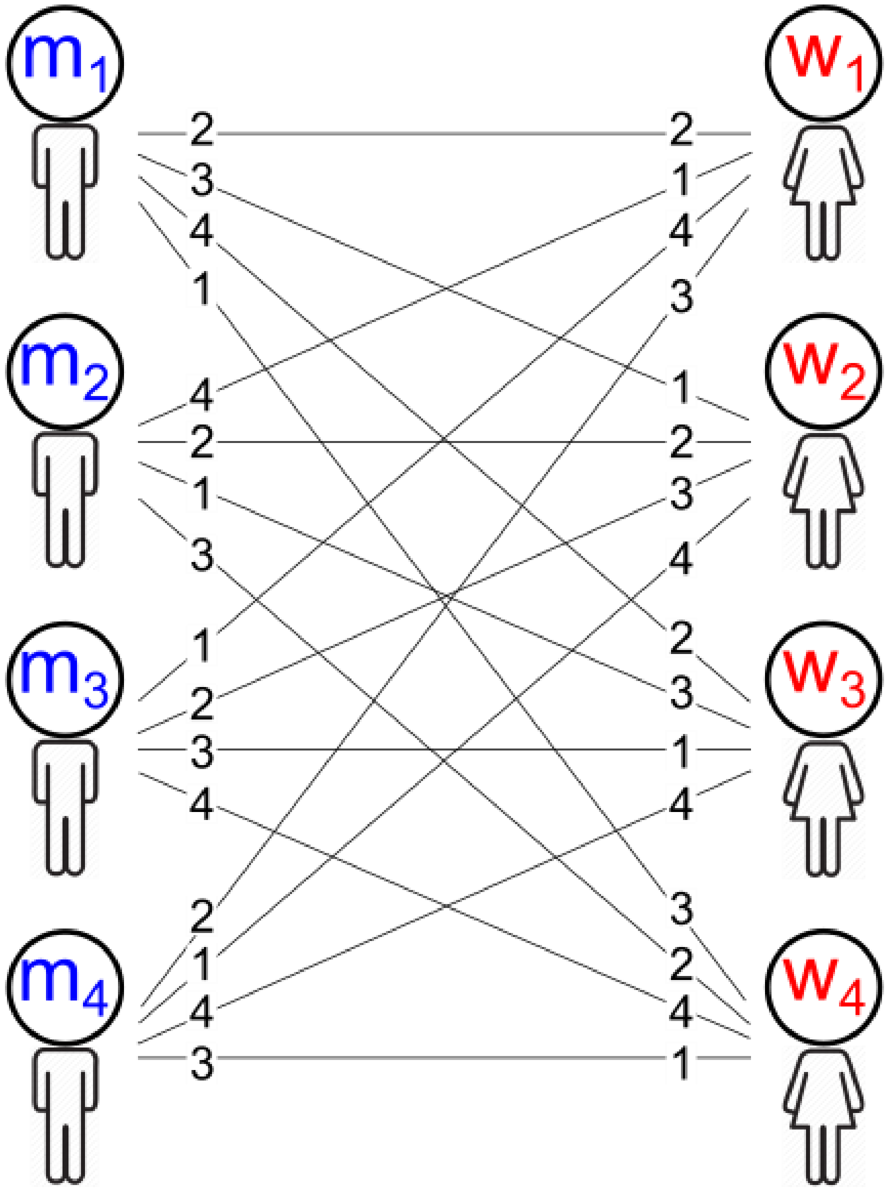
m1
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w2
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w4
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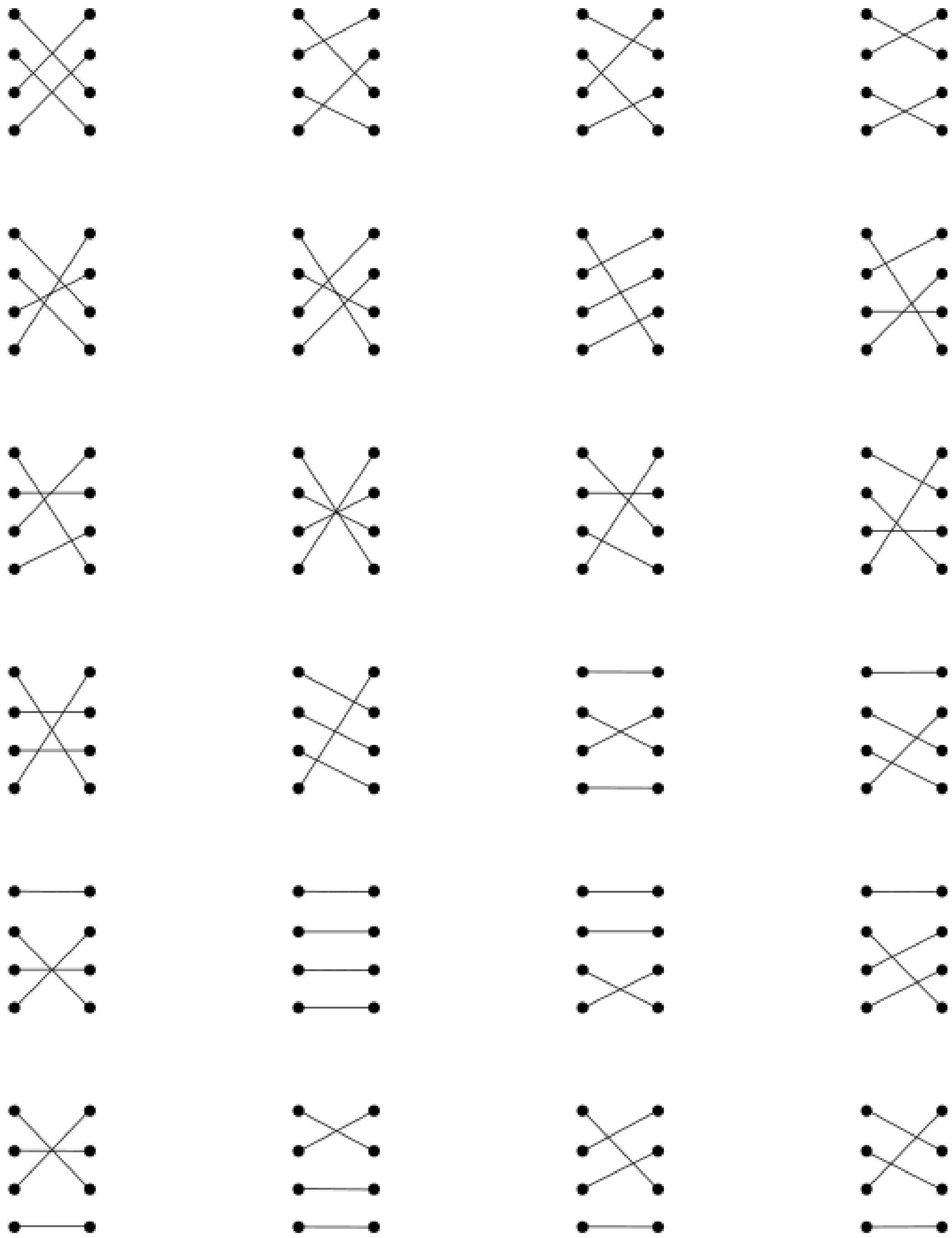
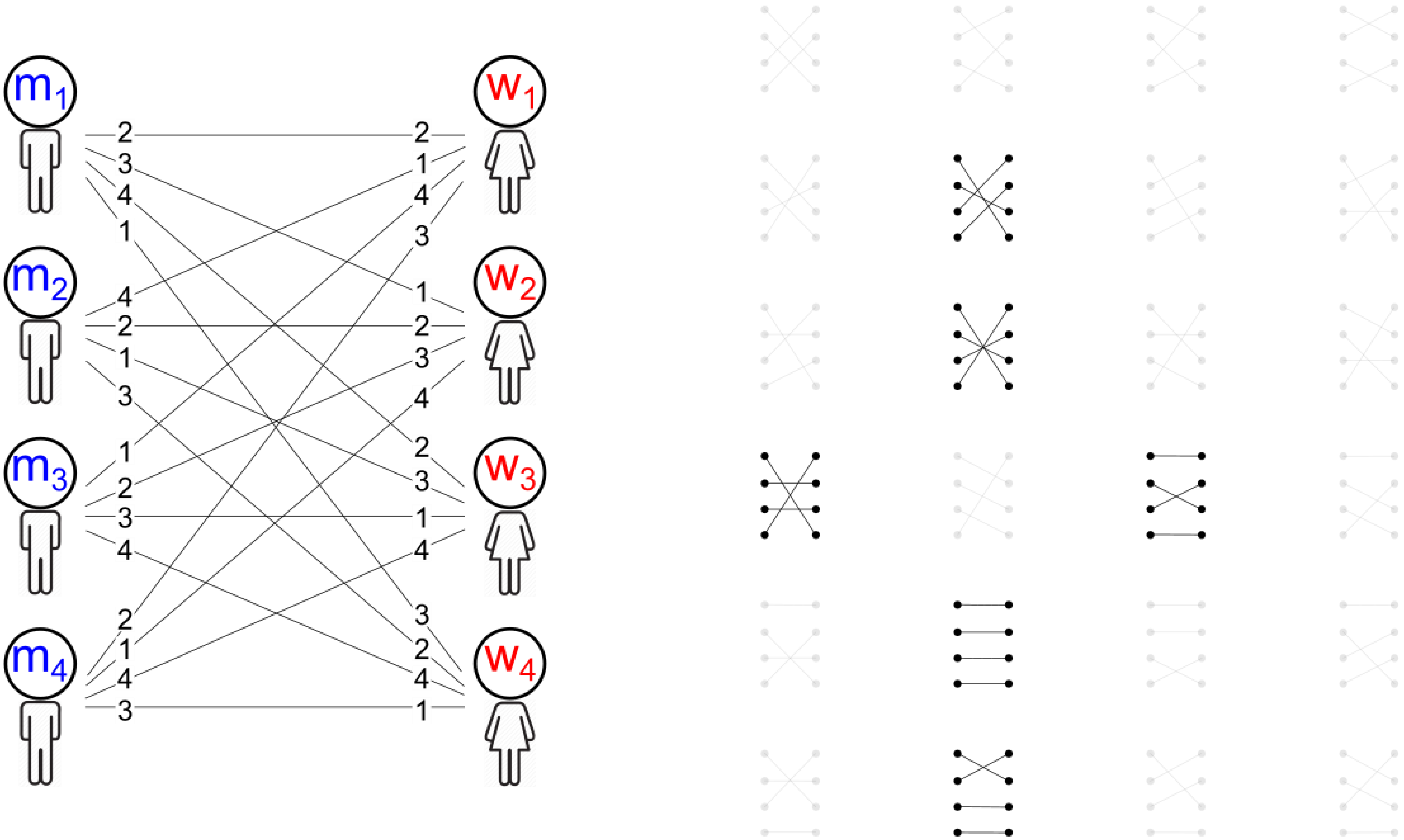
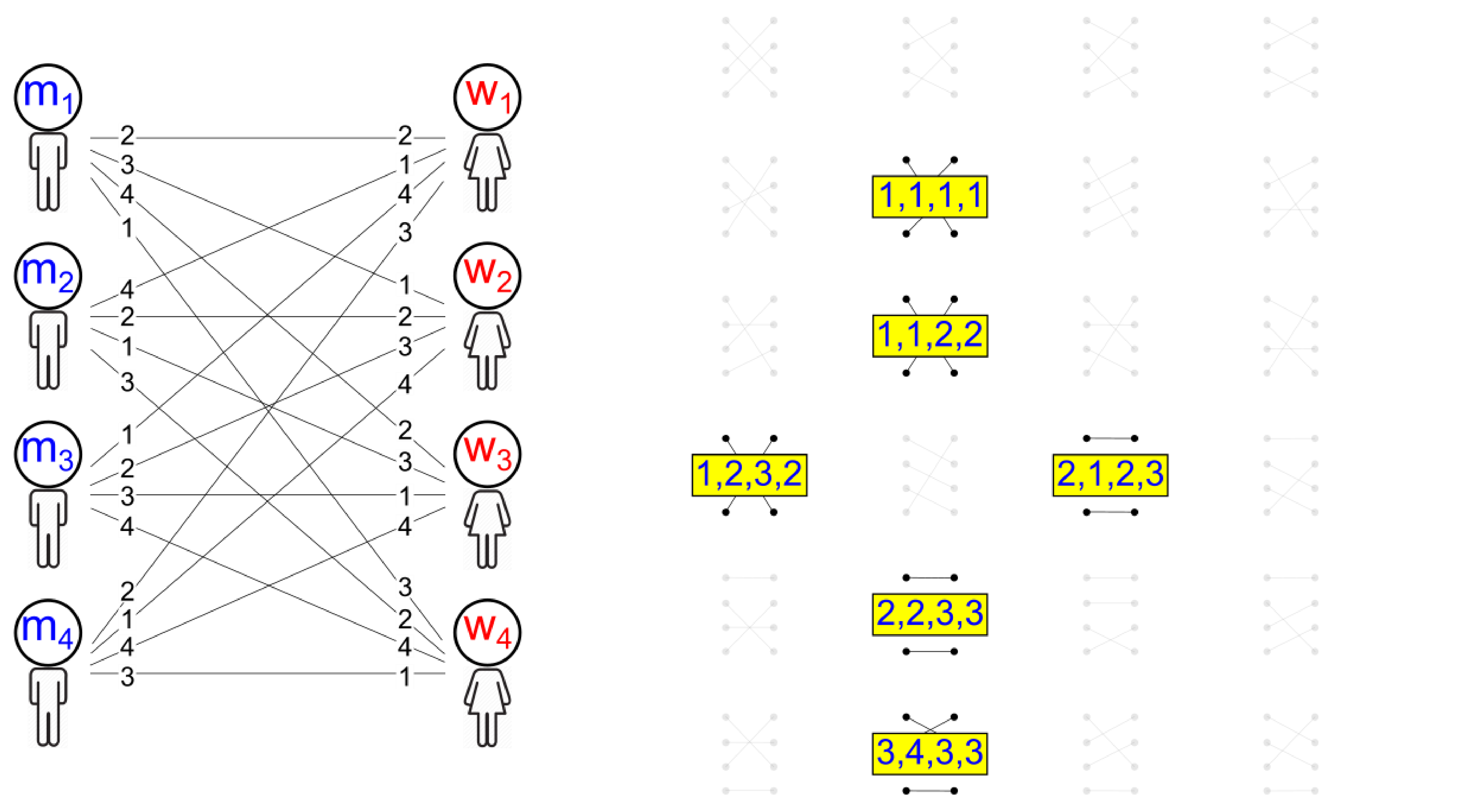
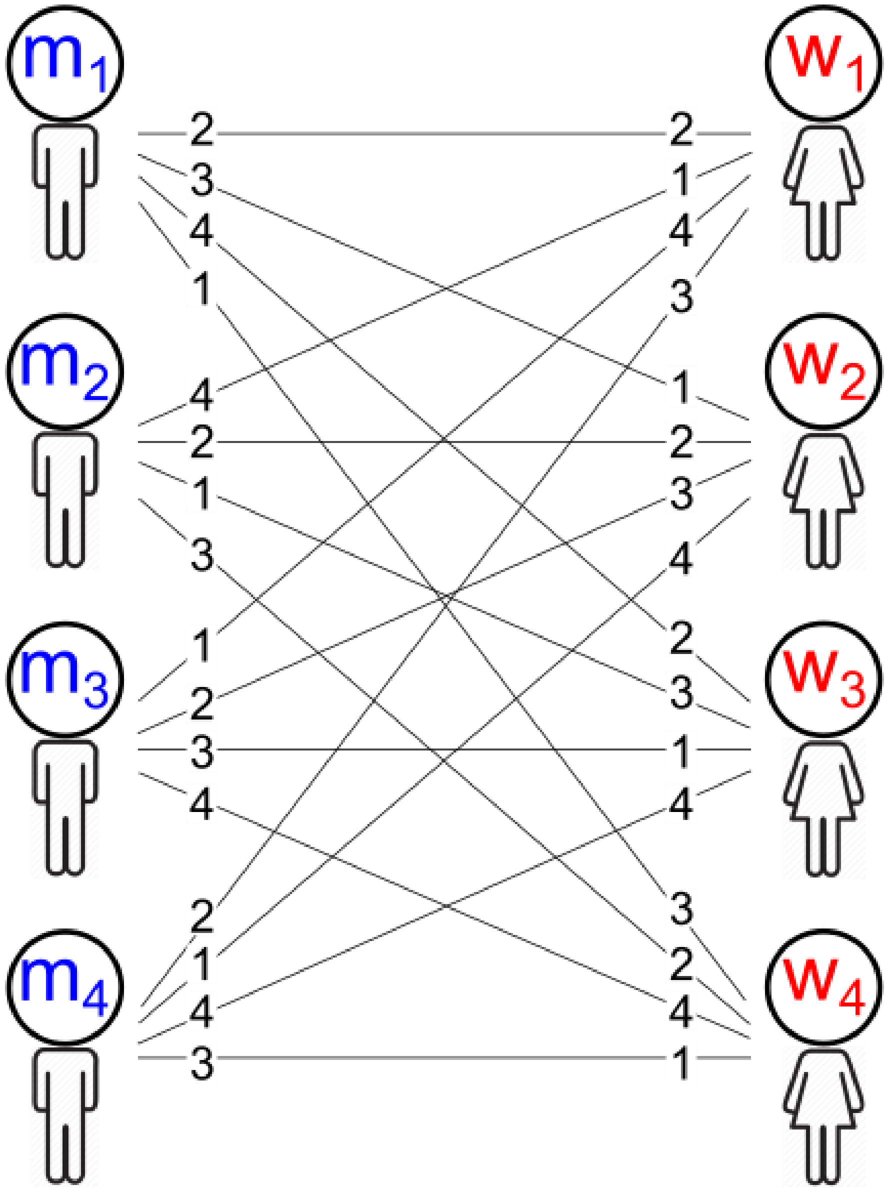




m1
m2
m3
m4
w1
w2
w3
w4


m1
m2
m3
m4
w1
w2
w3
w4
1,1,1,1
1,1,2,2
1,2,3,2
2,1,2,3
2,2,3,3
3,4,3,3



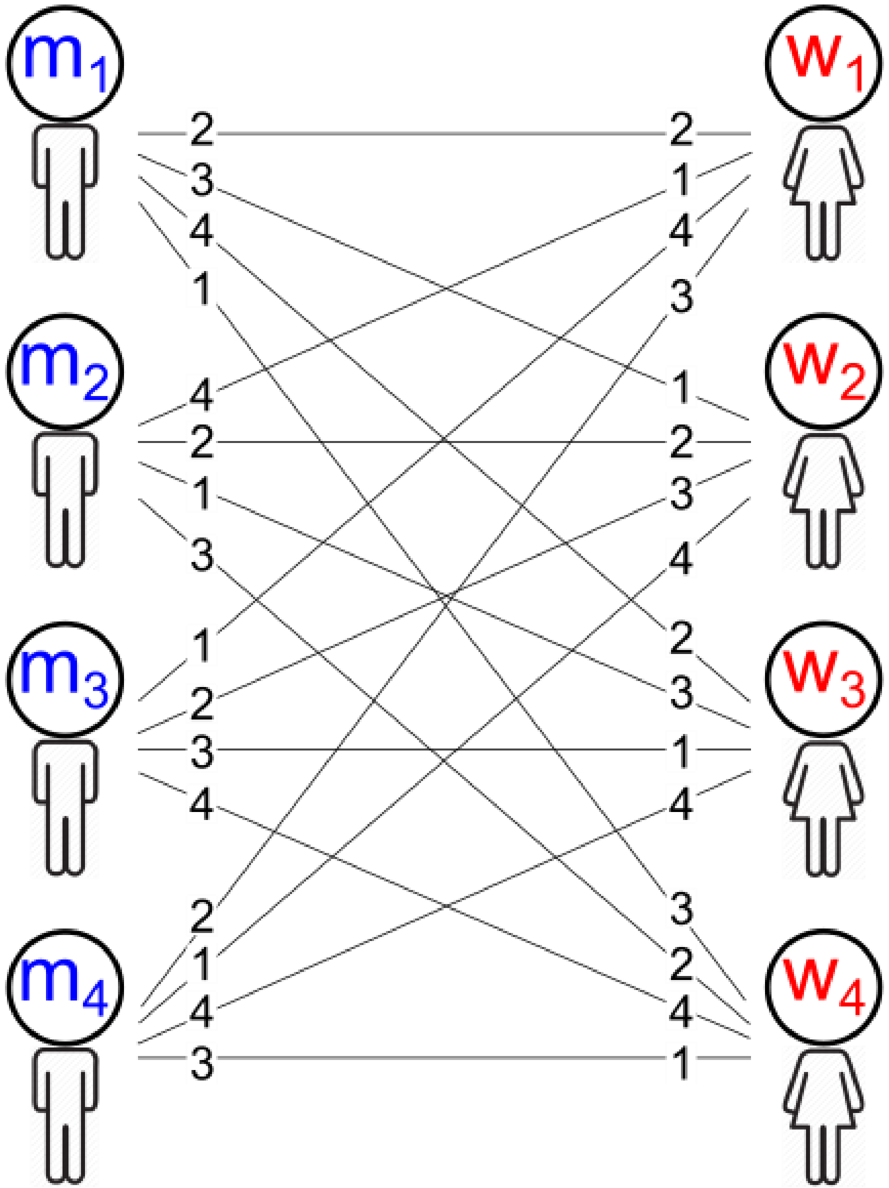
1,1,1,1

1,1,2,2

1,2,3,2 2,1,2,3

2,2,3,3

3,4,3,3



Men-optimal 1,1,1,1

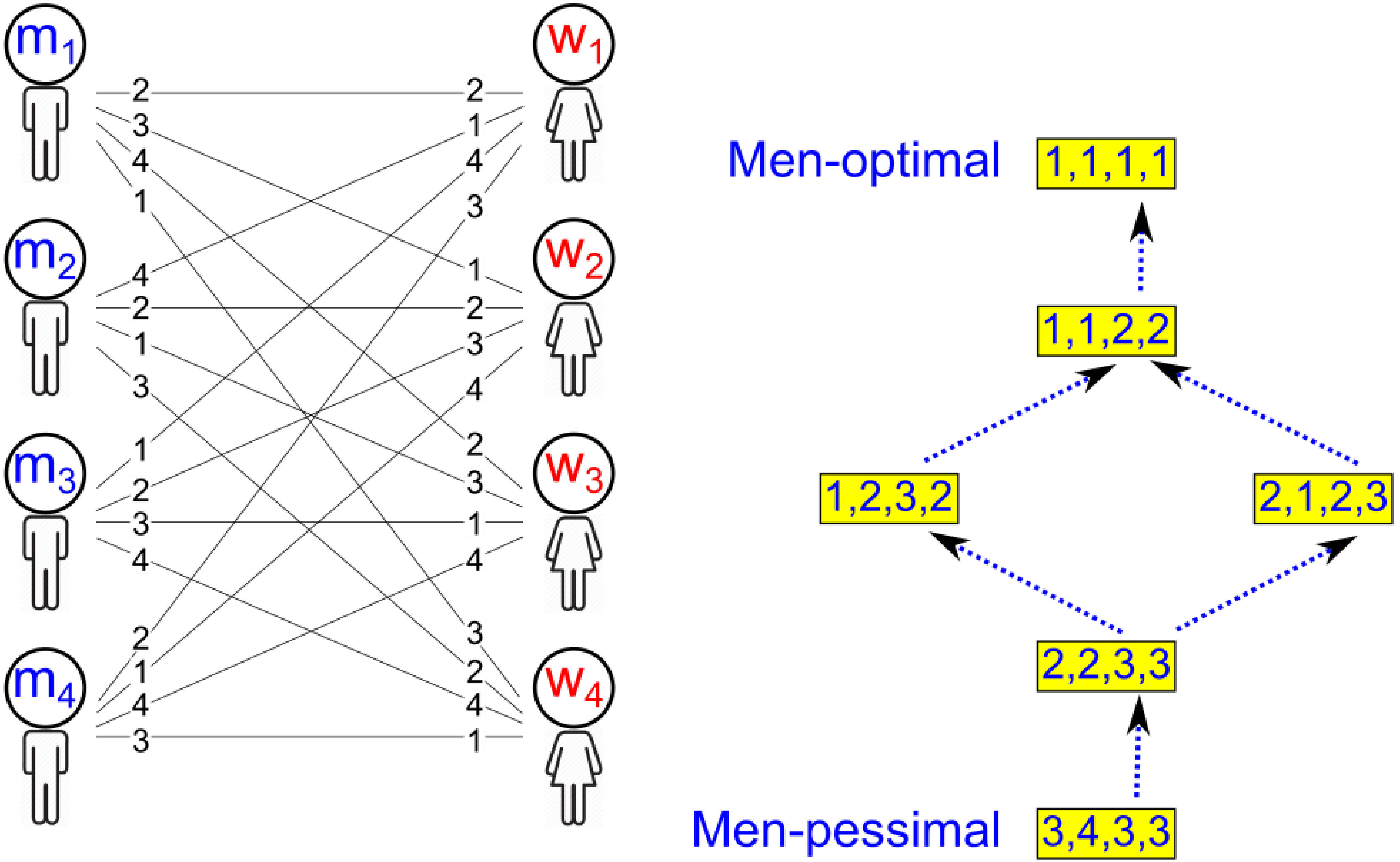
1,1,2,2

1,2,3,2

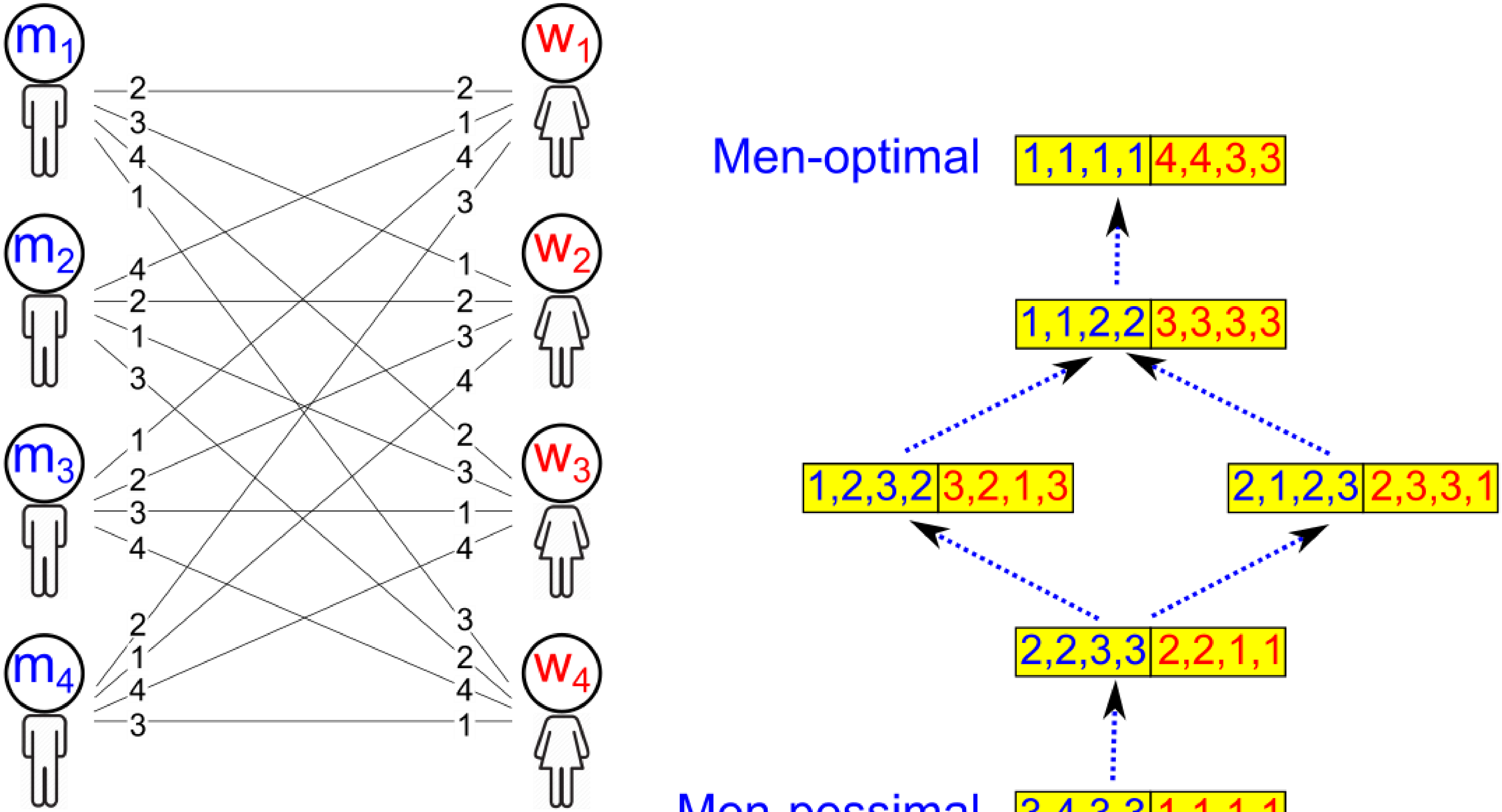
2,1,2,3

2,2,3,3

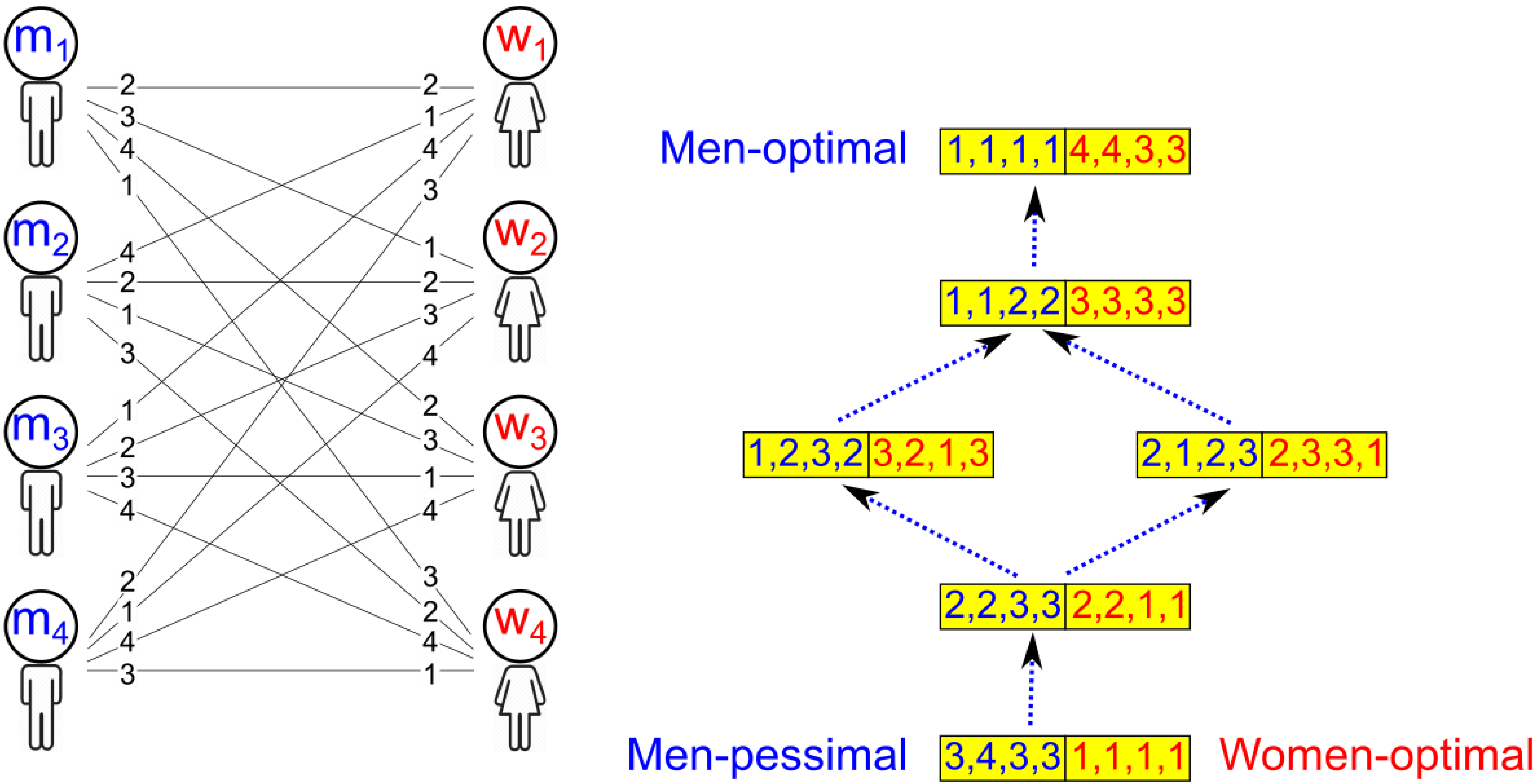
Men-pessimal 3,4,3,3



m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
1 2 3 4
2 3 1 4
3 2 4 1
Men-optimal
1,1,1,1
1,1,2,2
1,2,3,2
2,1,2,3
2,2,3,3
Men-pessimal
3,4,3,3



m1
m2
m3
m4
w1
w2
w3
w4
Men-optimal
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1


m_1
m_2
m_3
m_4
w_1
w_2
w_3
w_4
Men-optimal
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
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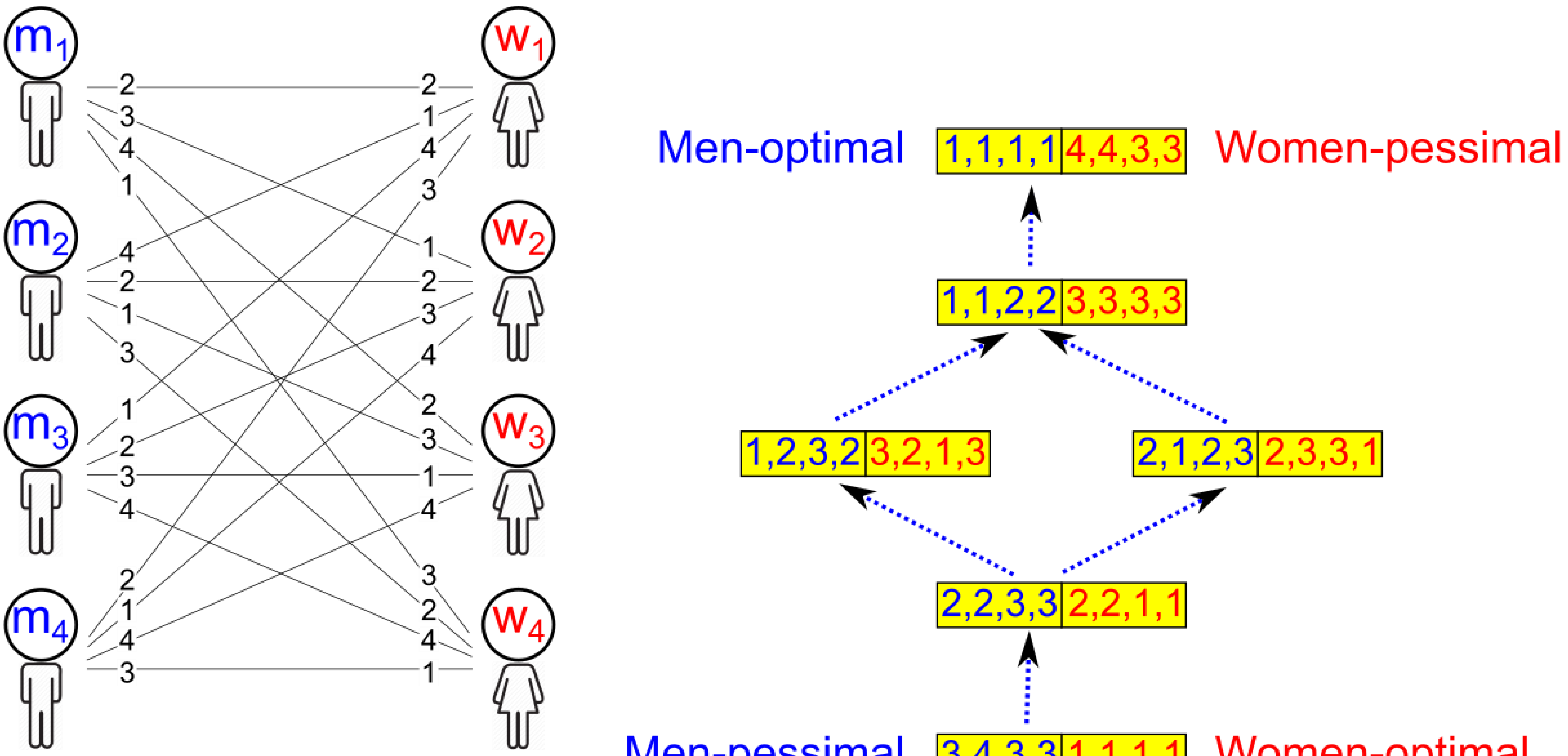
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal



m1
m2
m3
m4
w1
w2
w3
w4
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
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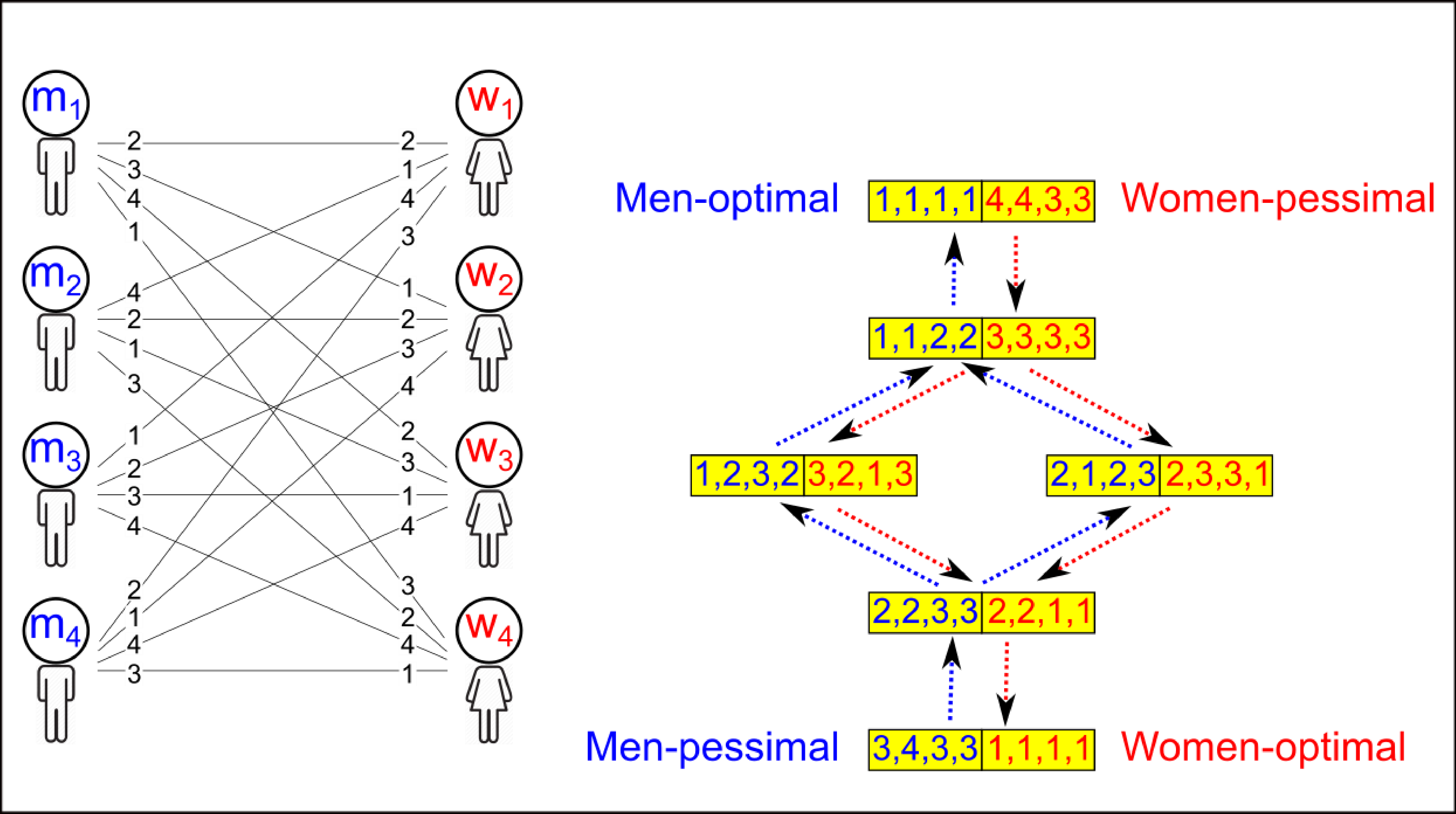
m1
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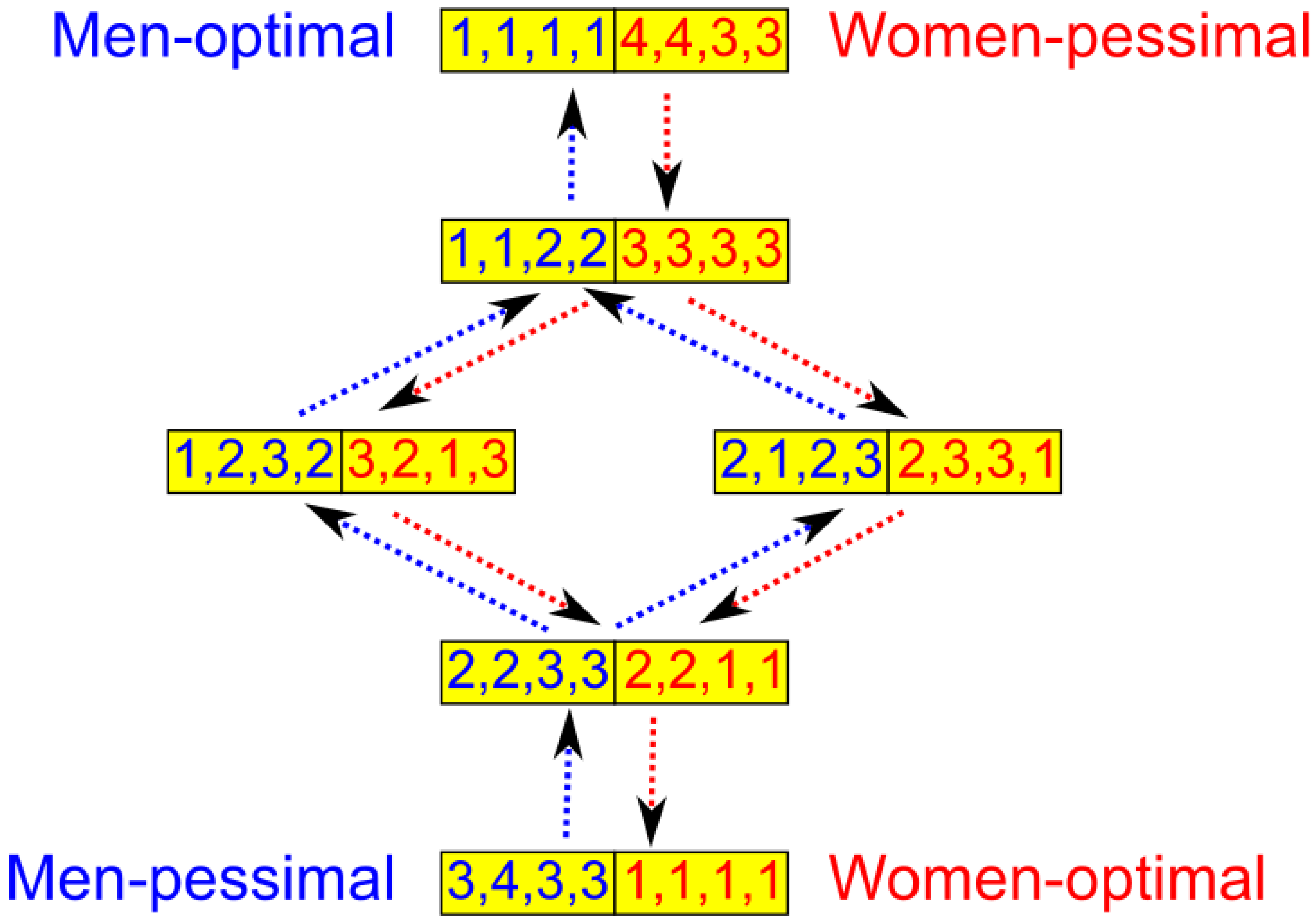
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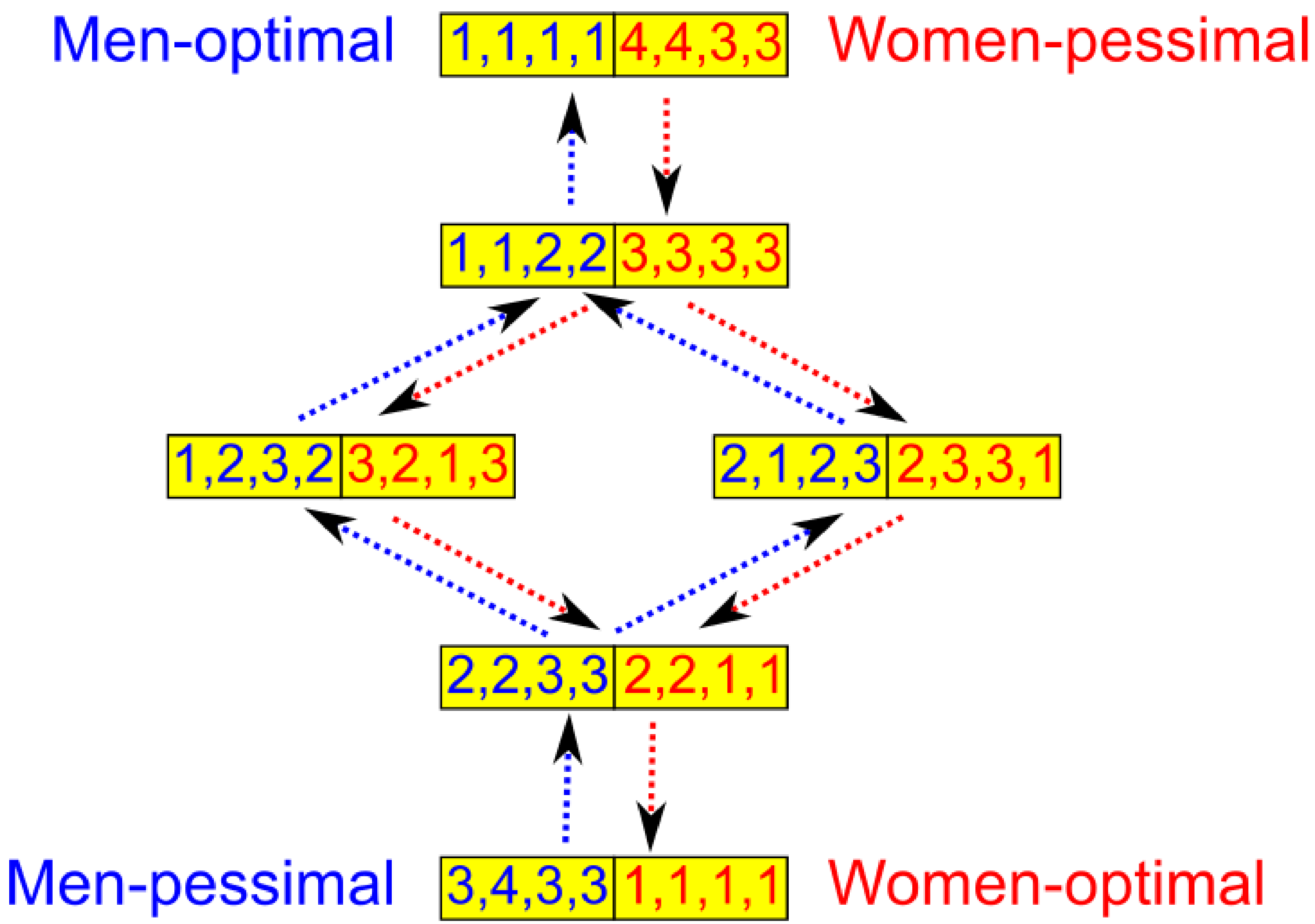
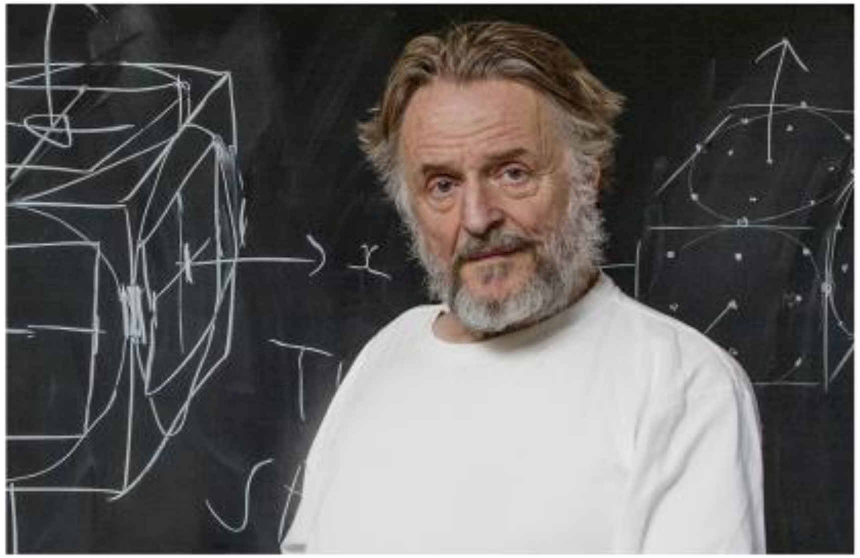
The Lattice of Stable Matchings





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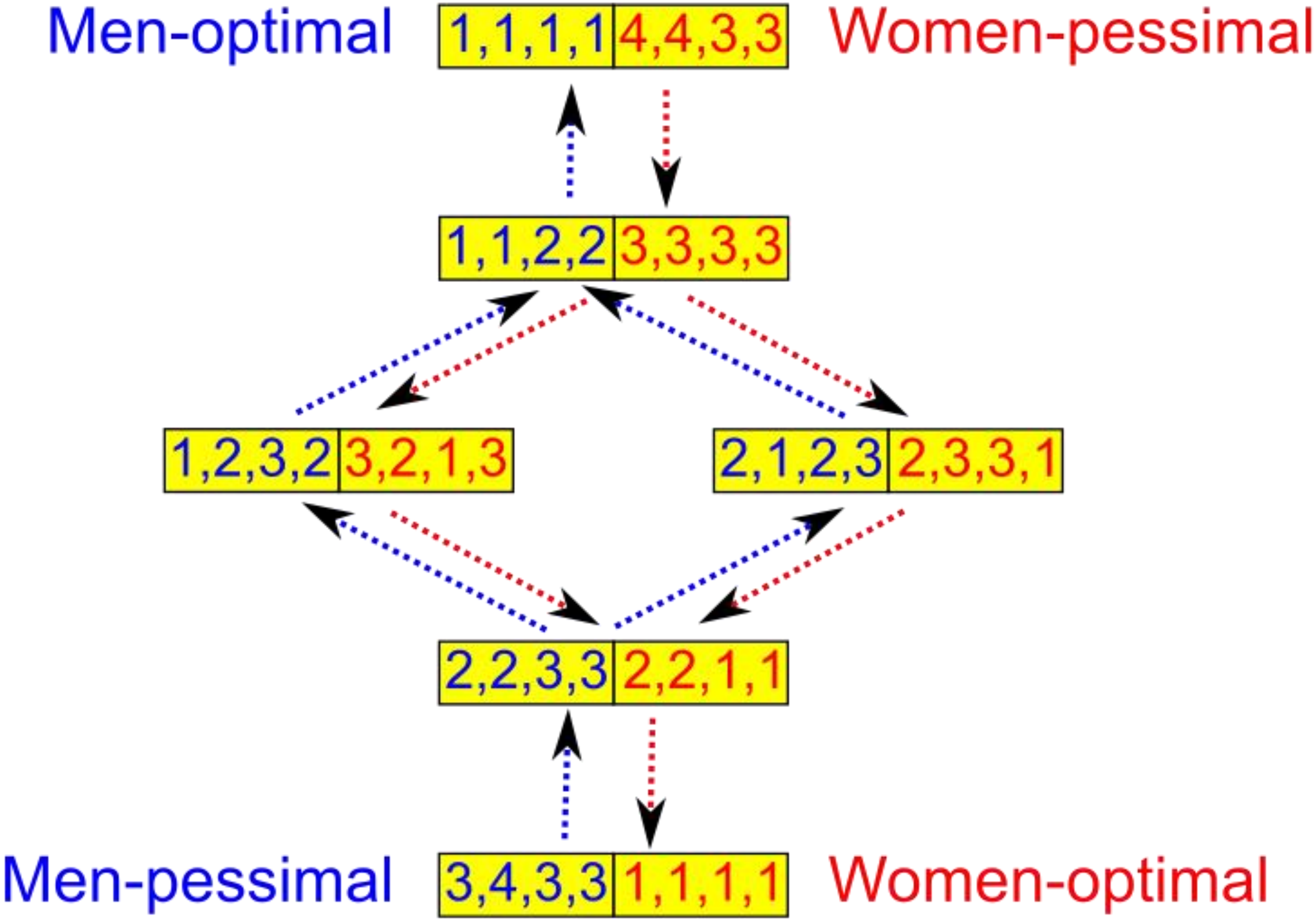

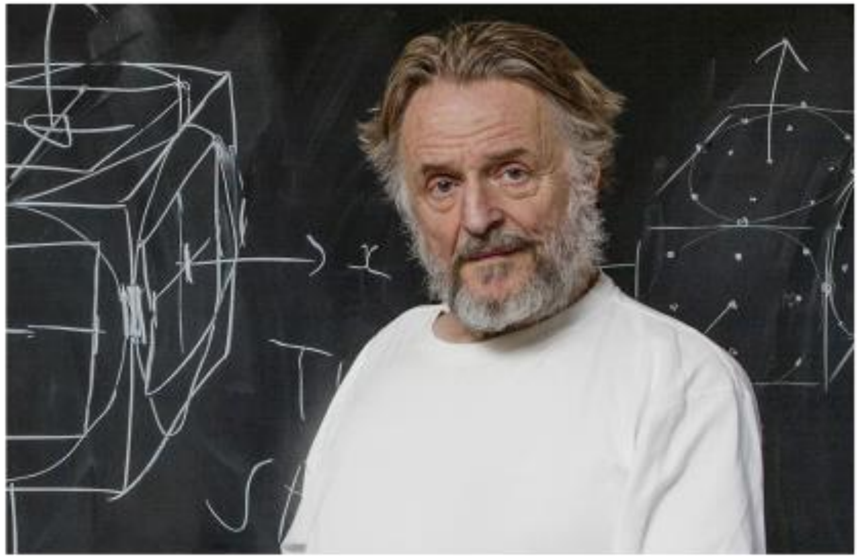

John H. Conway

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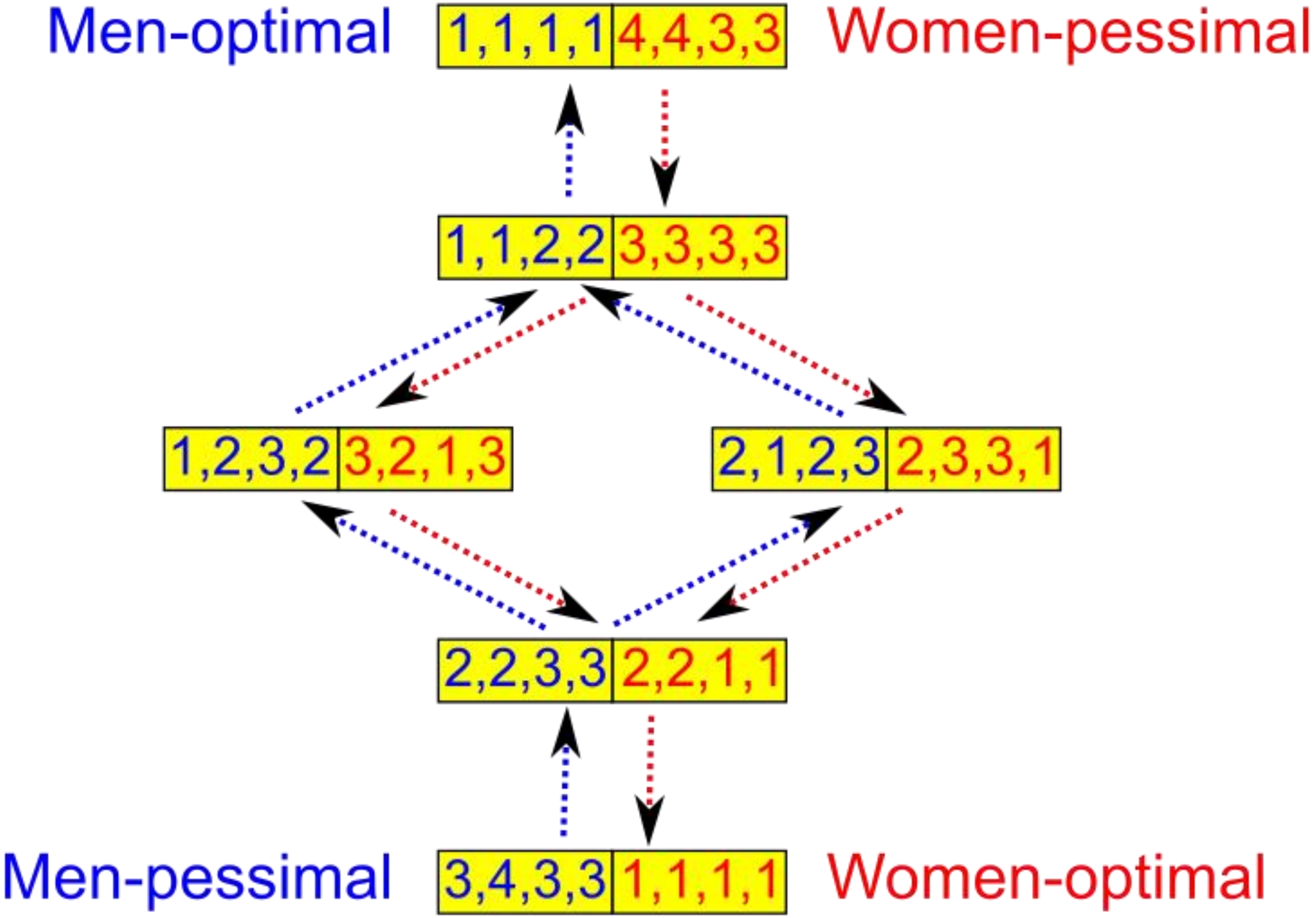

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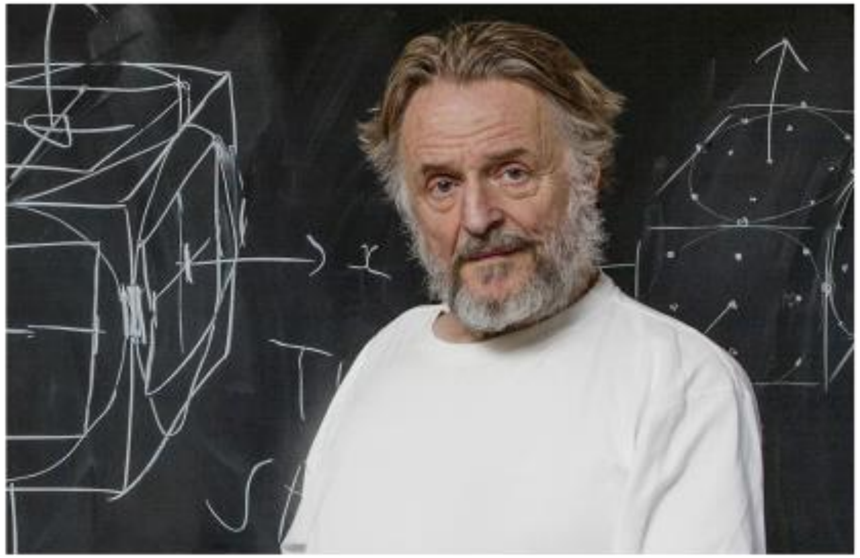

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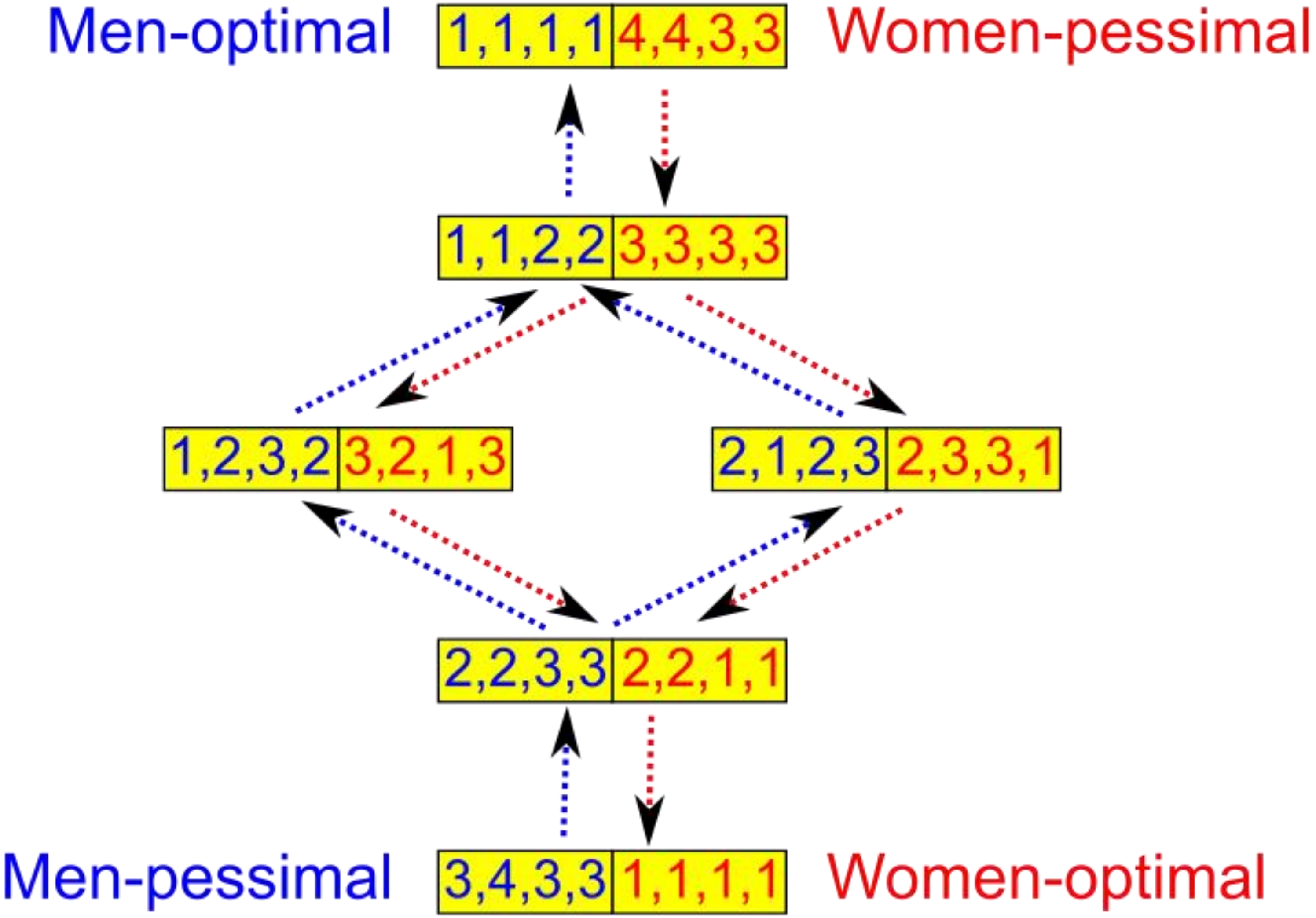

Some Observations

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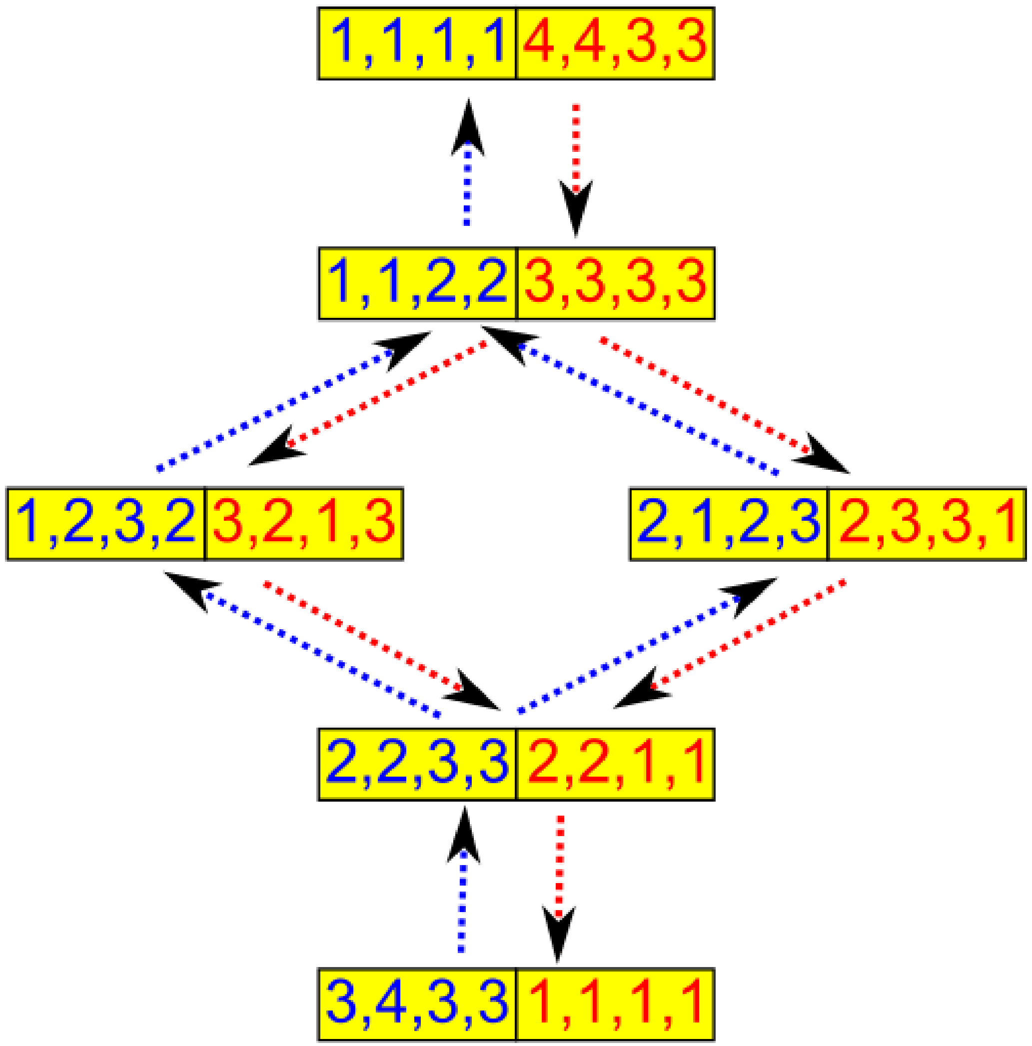

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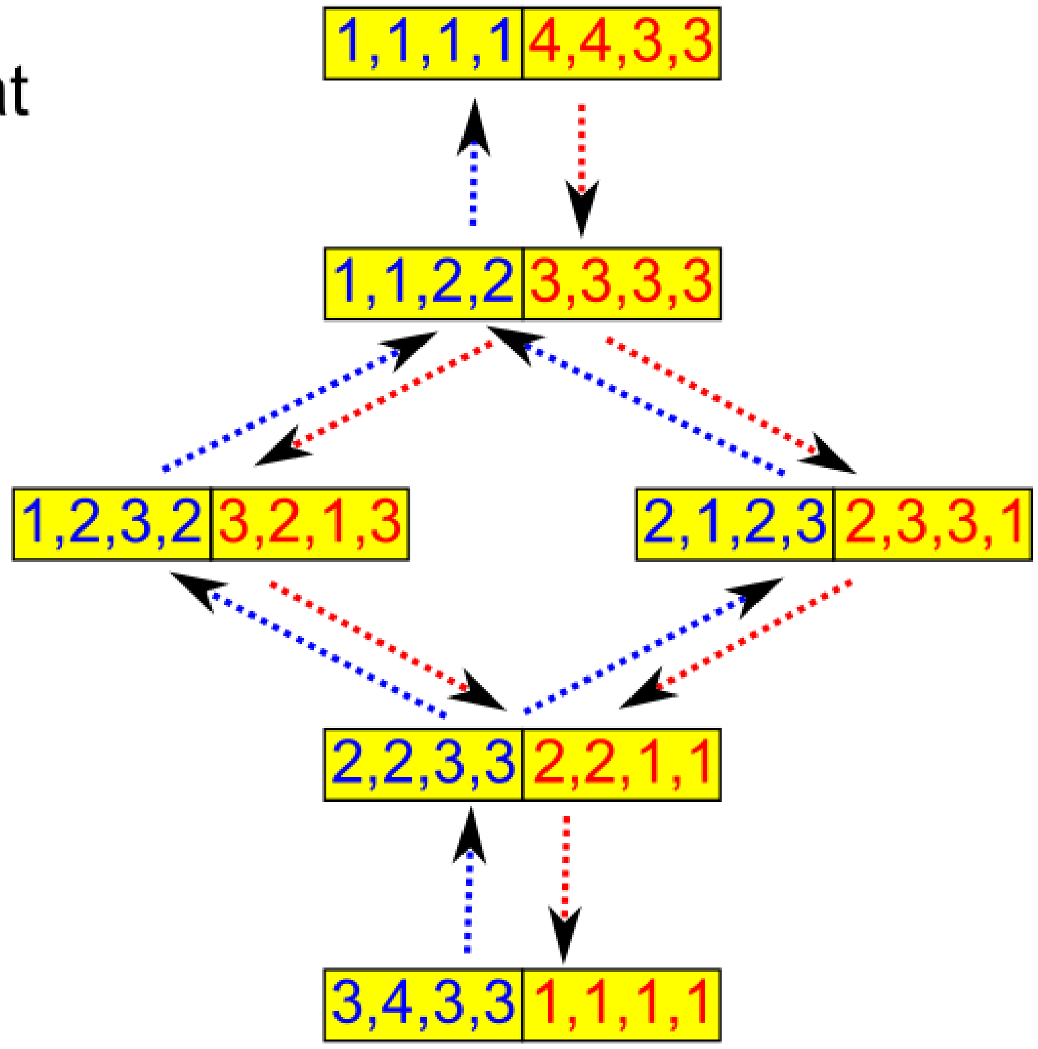

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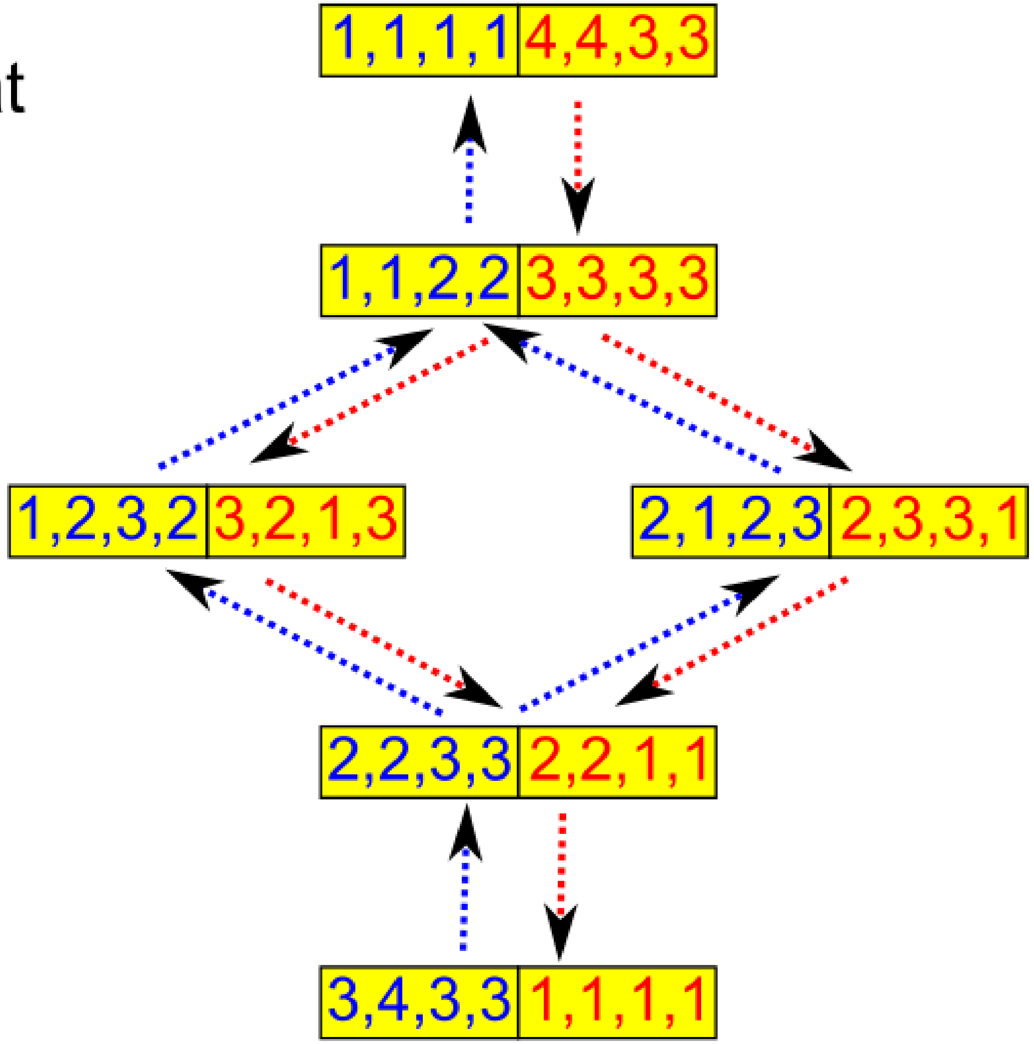

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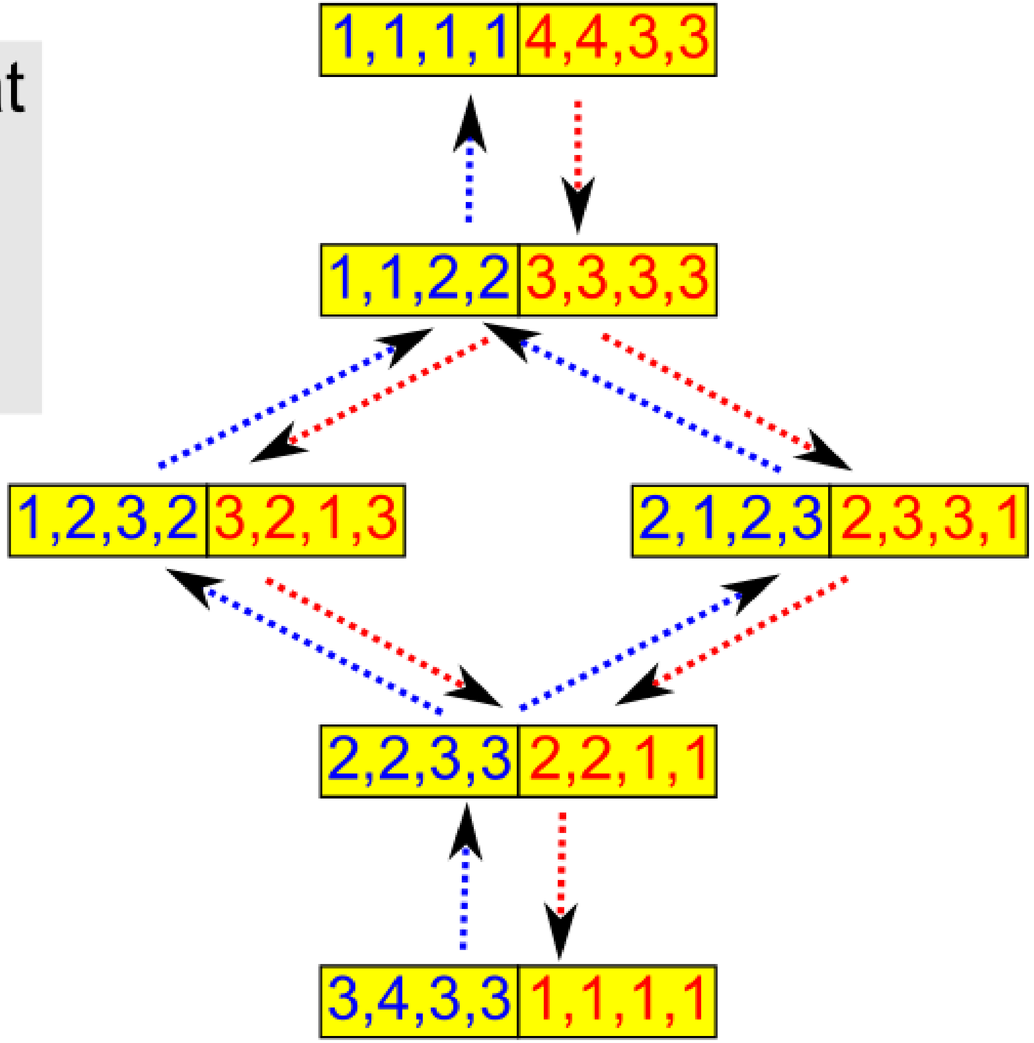

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We will show that men/women-optimal mappings are actually *matchings*.

[Gale and Shapley, 1962]

Given any preference profile, the matching computed by the men-proposing deferred-acceptance algorithm is men-optimal. Similarly, a women-optimal matching is obtained when women propose.

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When m' proposes to w, his past rejections (if any) must all have been from women that are *unachievable* for him.

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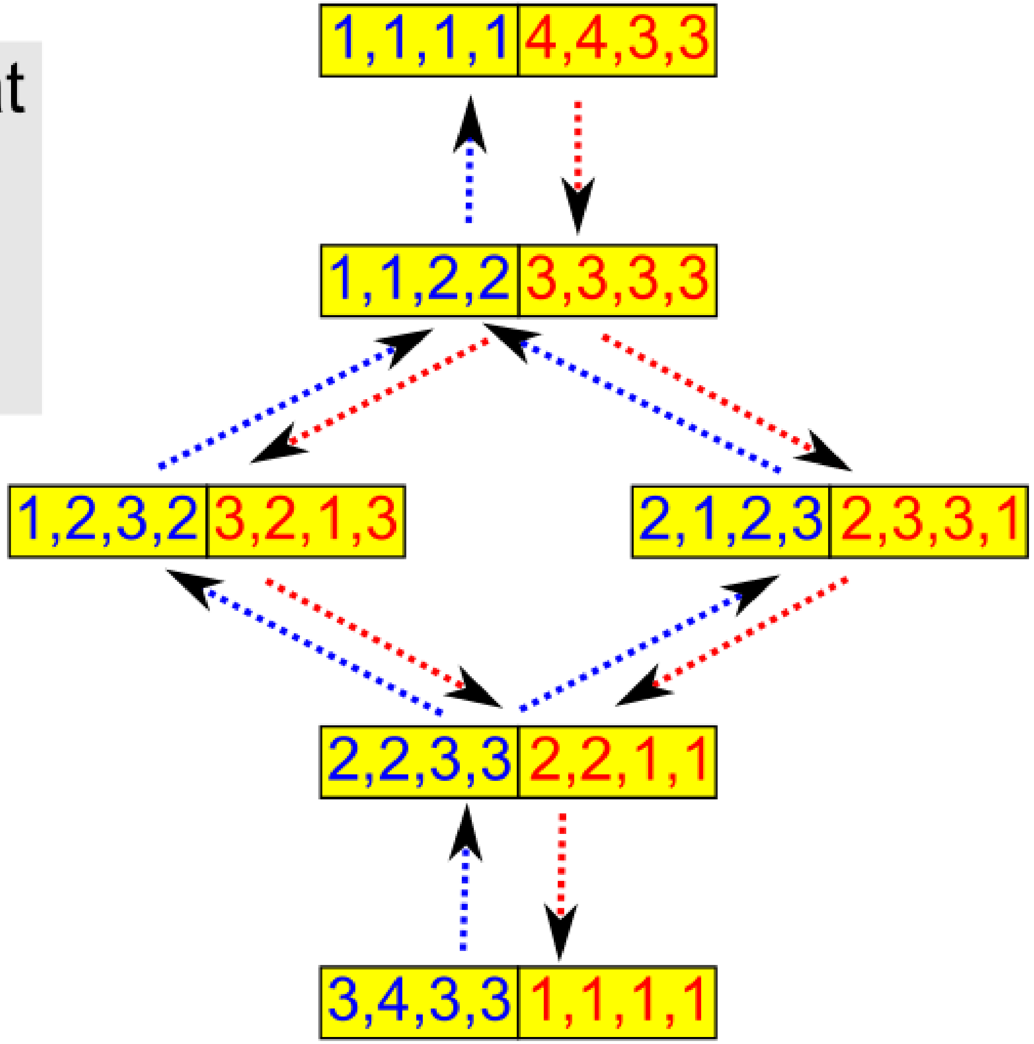

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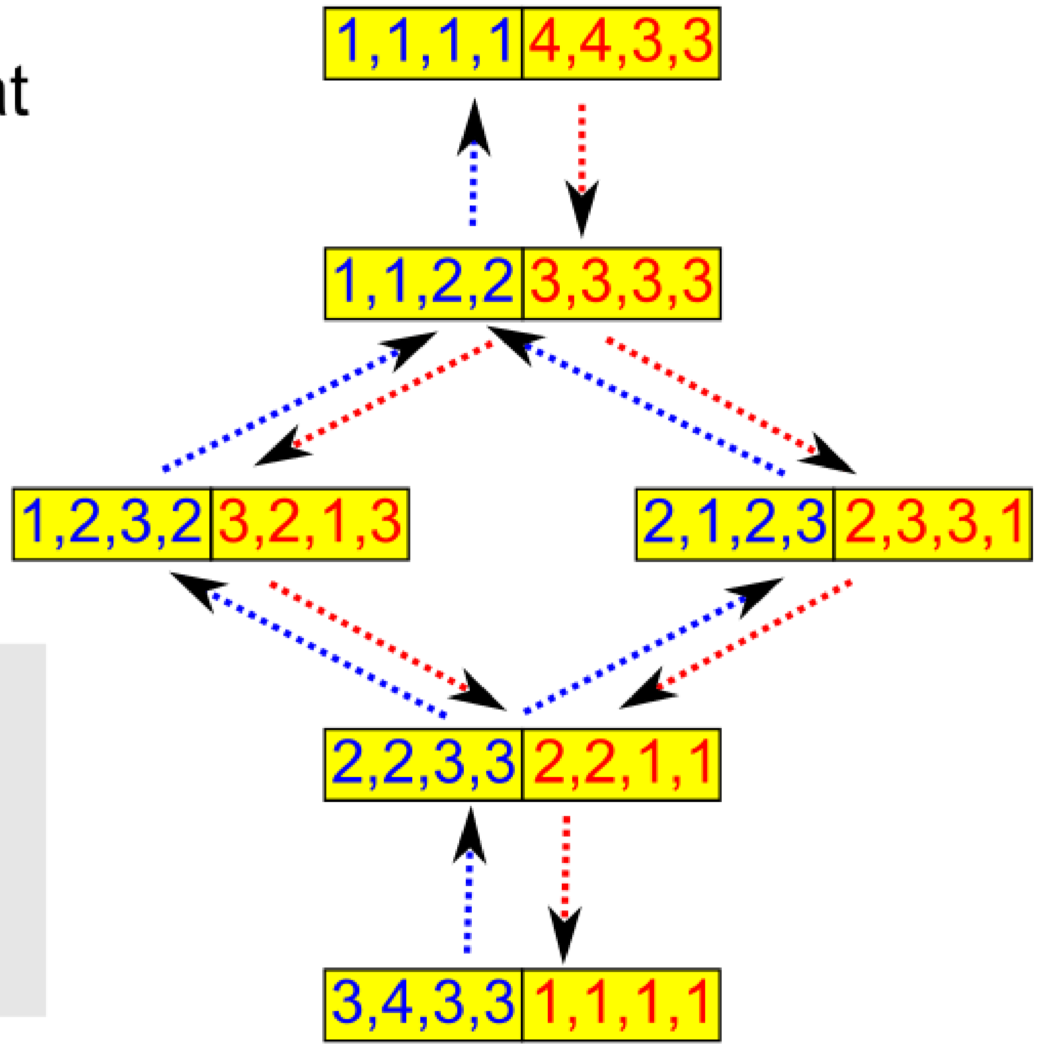

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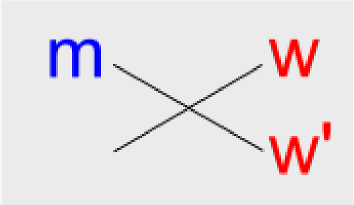

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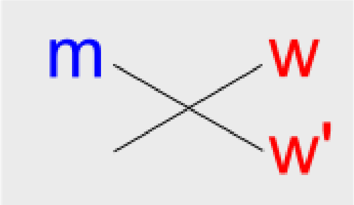

Then, m prefers w over w', and w prefers m over her Q-partner.

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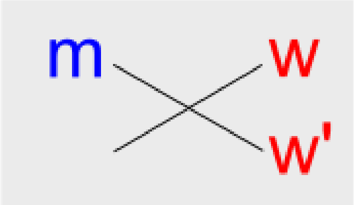
Q: 

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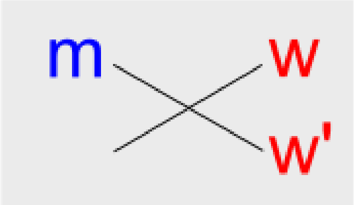
Q: 

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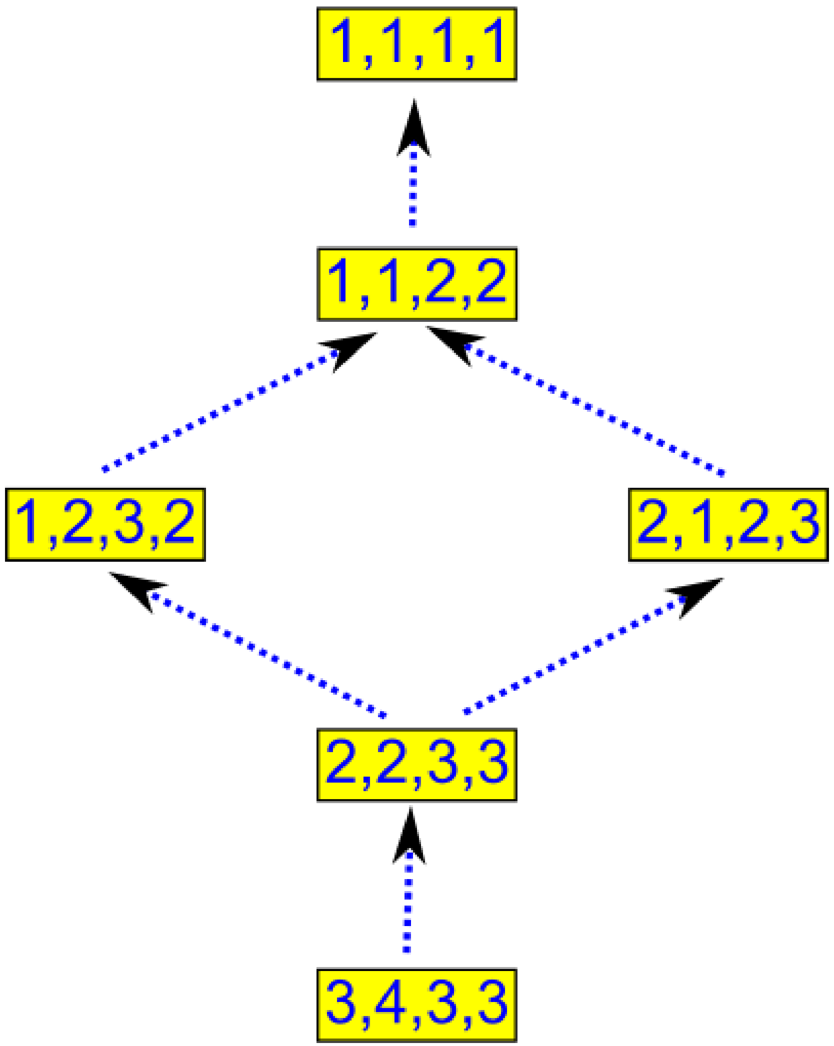

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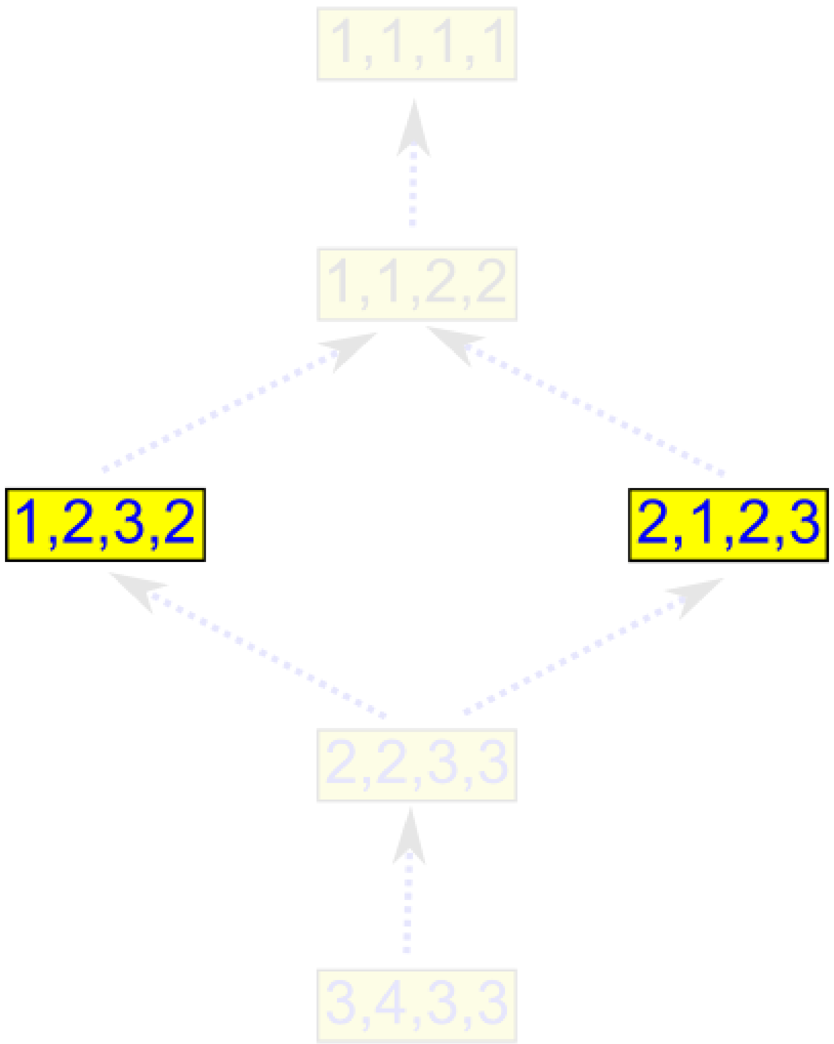

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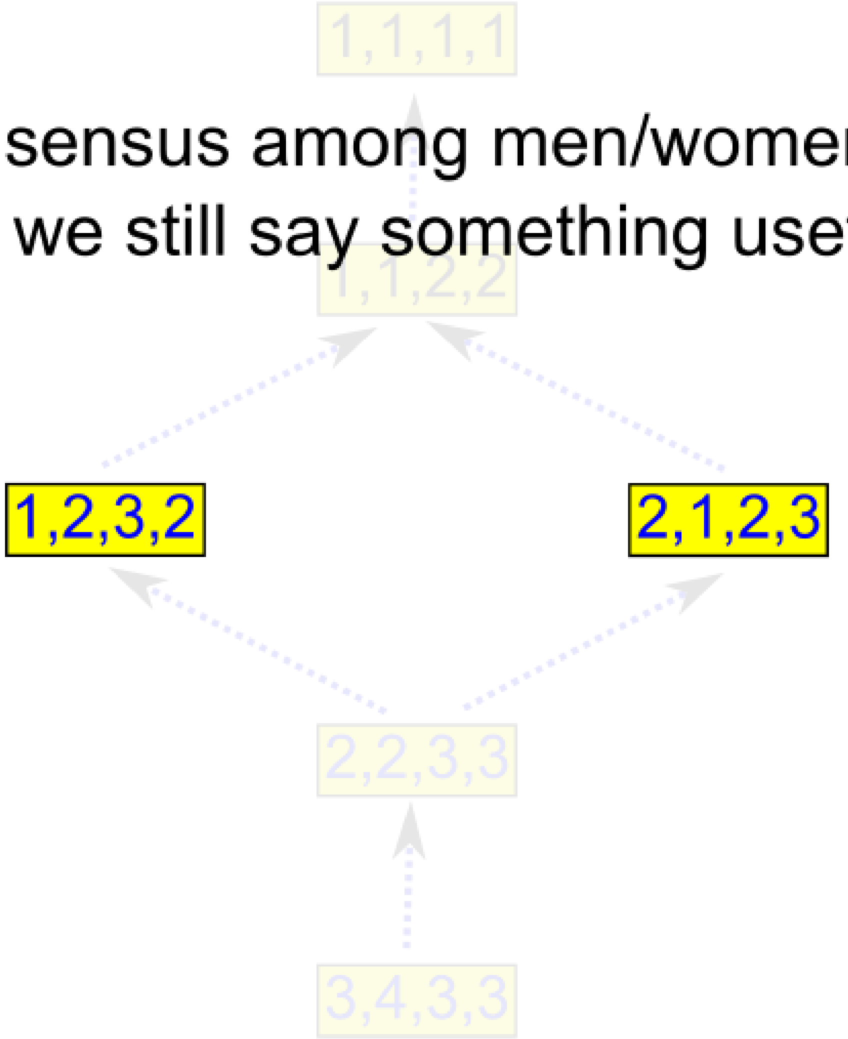
When there isn't a consensus among men/women w.r.t. two matchings, can we still say something useful?



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Let's generalize this idea to arbitrary pairs of stable matchings.

Let P and Q be any pair of stable matchings (not necessarily distinct).

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Define a *mapping* $\max_{P,Q}$ that maps:

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$$\max_{P,Q}(m) = \begin{cases} P(m) \text{ if } m \text{ prefers } P(m) \text{ over } Q(m) \\ Q(m) \text{ otherwise} \end{cases}$$

$$\max_{P,Q}(w) = \begin{cases} Q(w) \text{ if } w \text{ prefers } P(w) \text{ over } Q(w) \\ P(w) \text{ otherwise} \end{cases}$$

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Why is $\max_{P,Q}$ a valid matching?

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Suffices to show that for any m and w, $\max_{P,Q}(m) = w \Leftrightarrow \max_{P,Q}(w) = m$.

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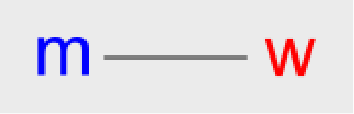
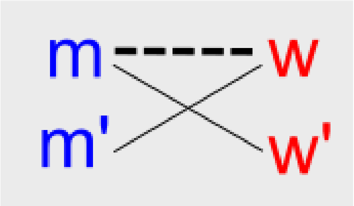
P: 

Q: 

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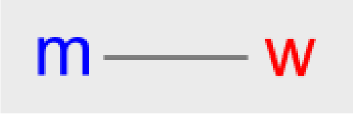

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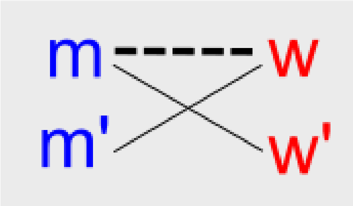

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"⇐" For every man m, there exists a woman w such that $max_{P,Q}(w) = m$.

If not, then consider the matching, say P, that m likes less. Man m and his preferred woman $max_{P,Q}(m)$ will block P, contradicting its stability.

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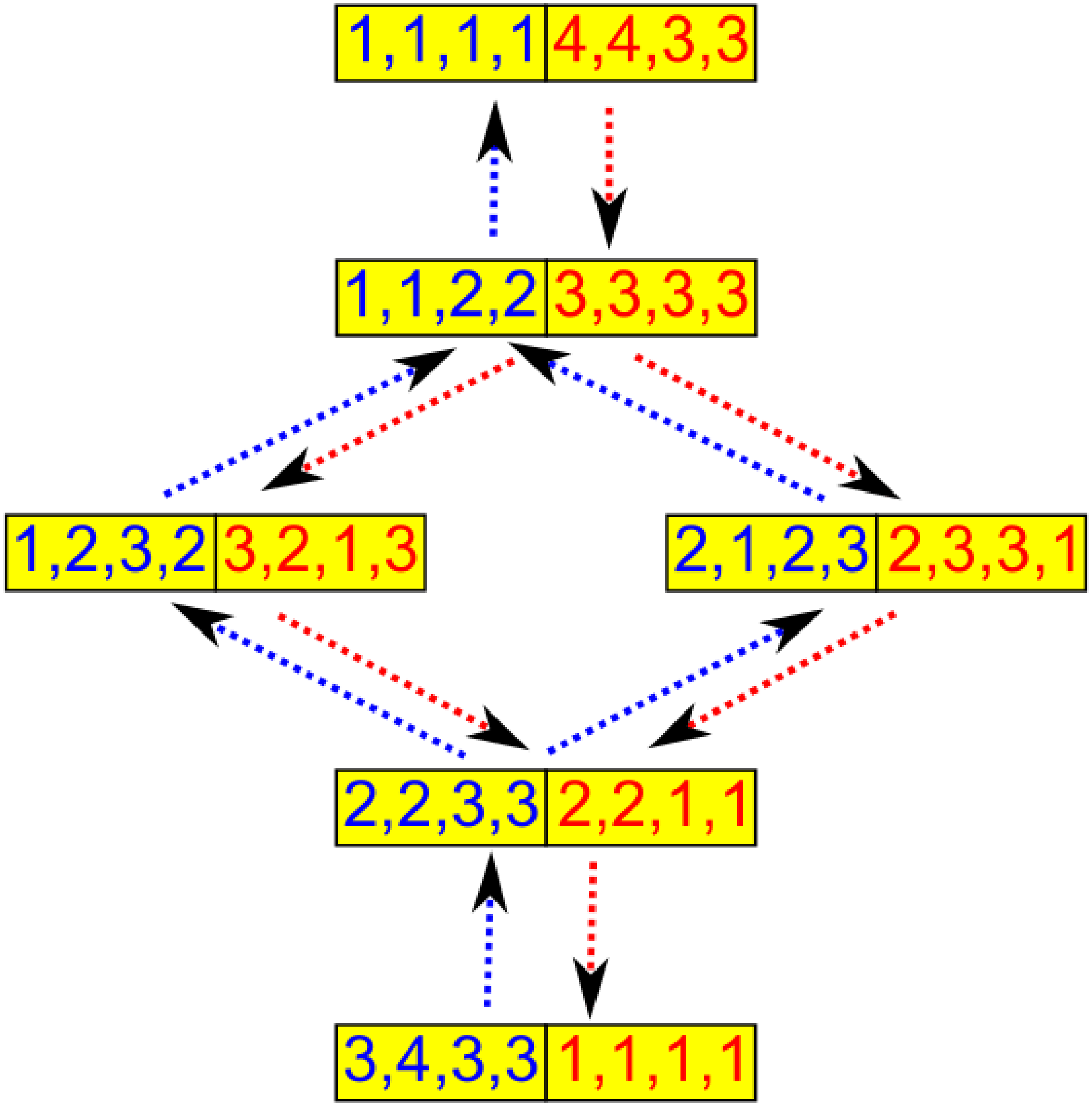
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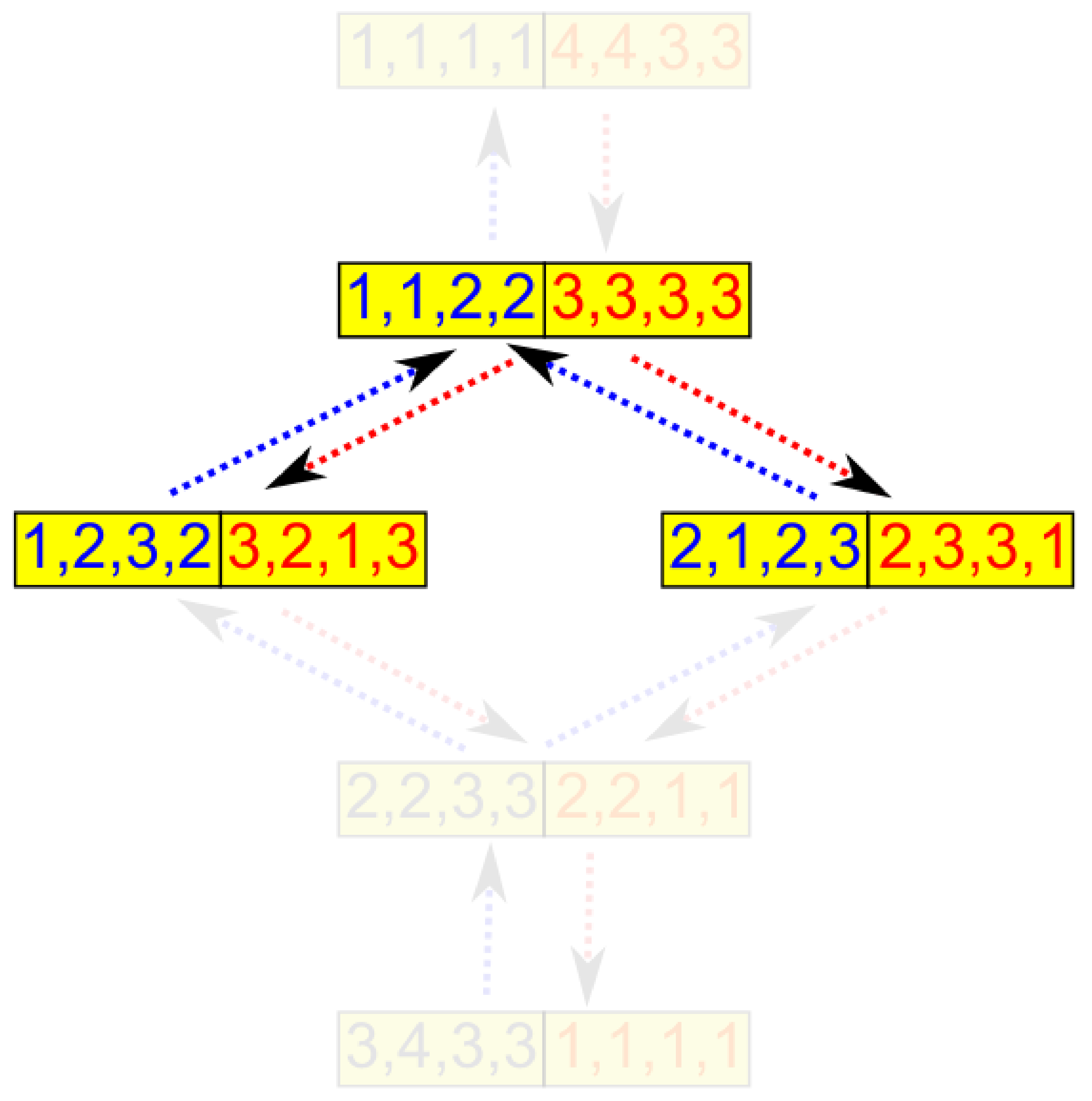

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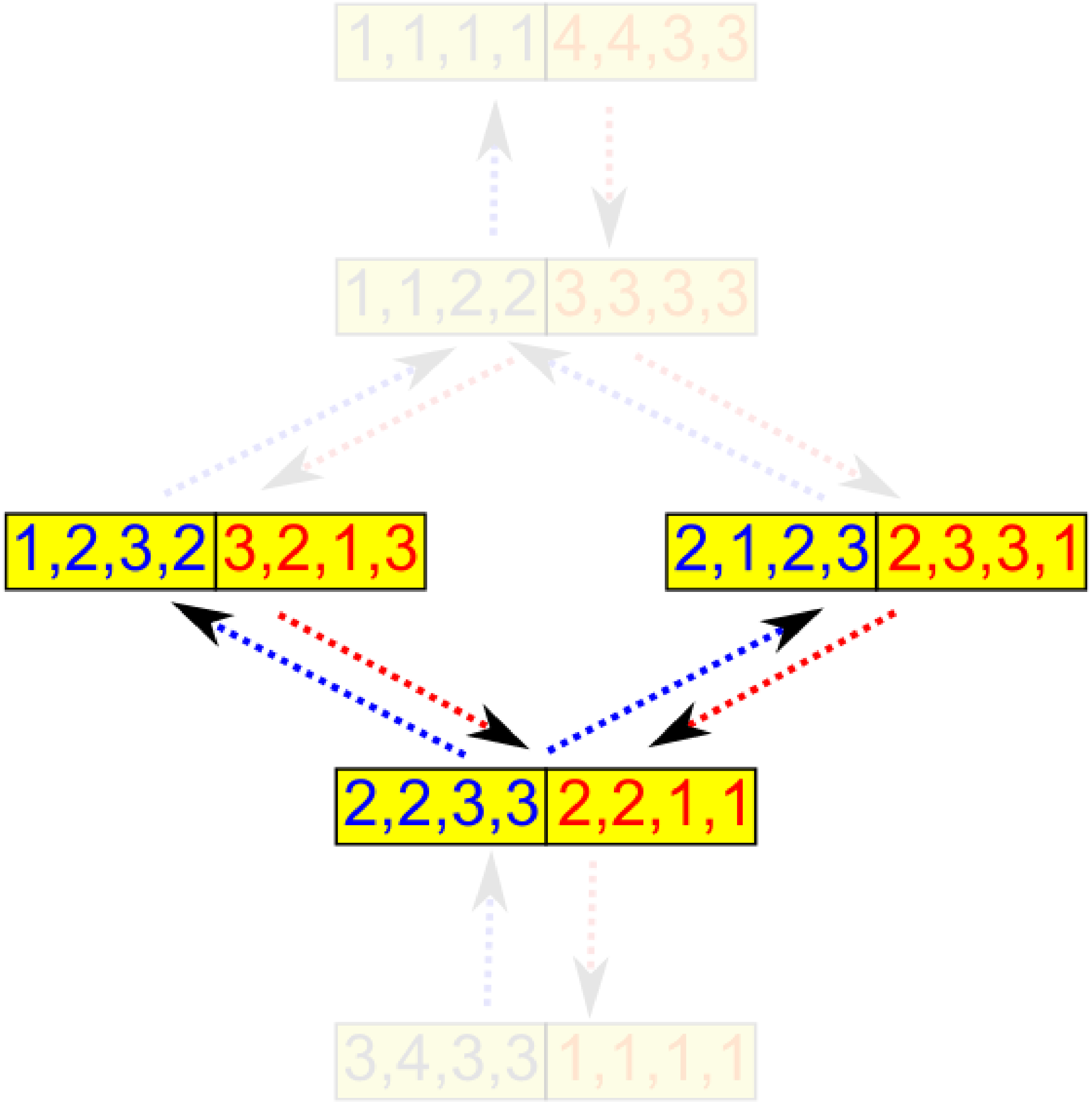
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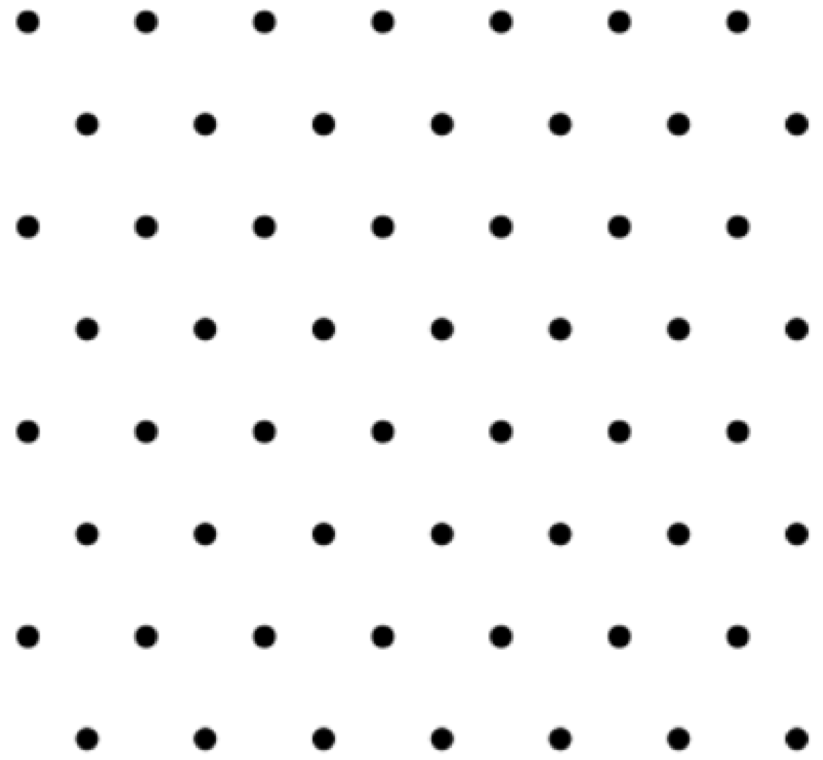

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Consequences:

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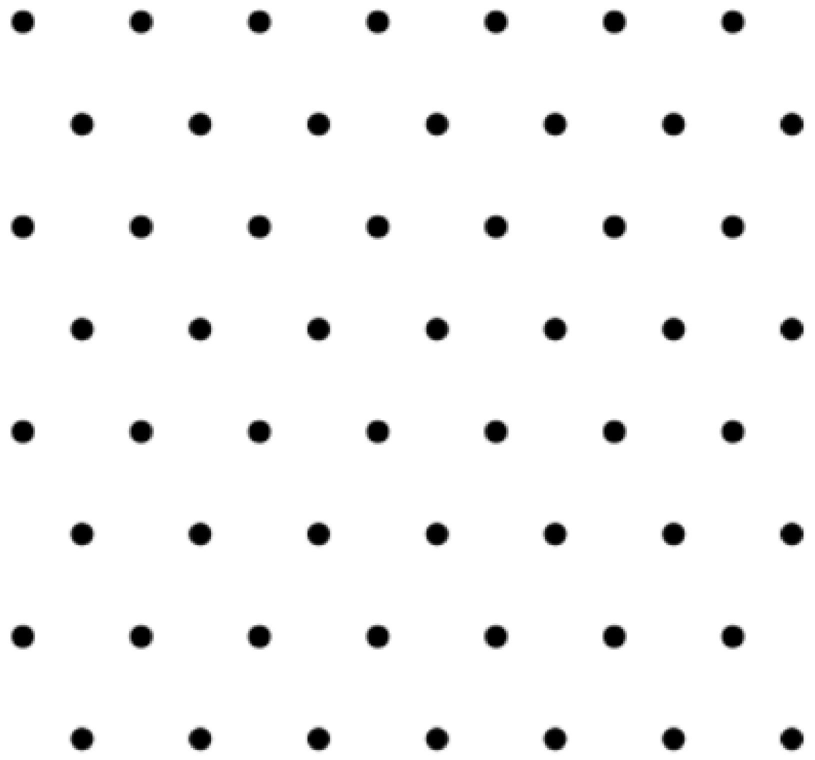

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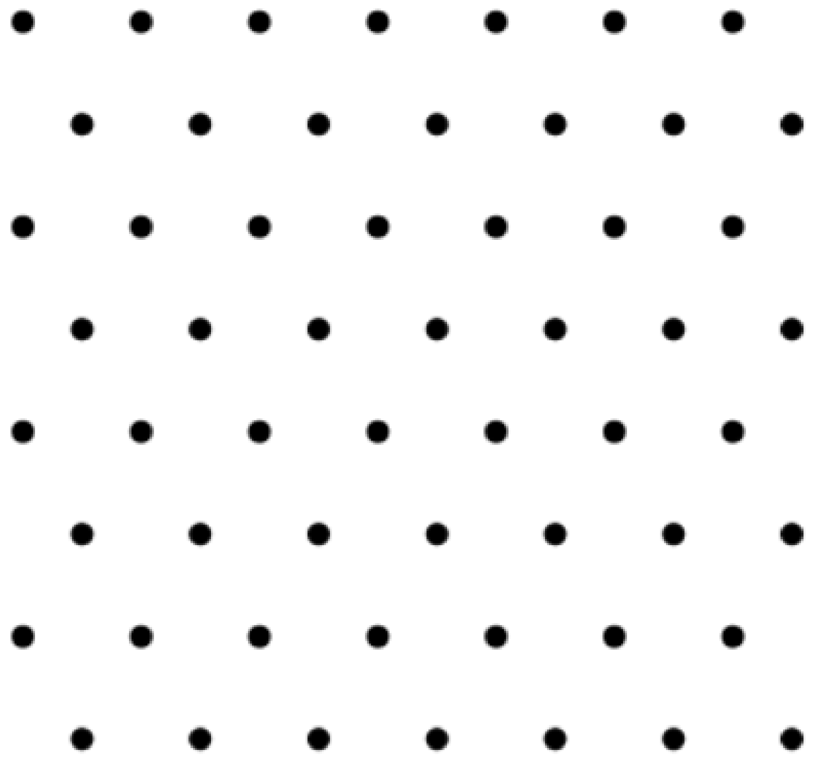

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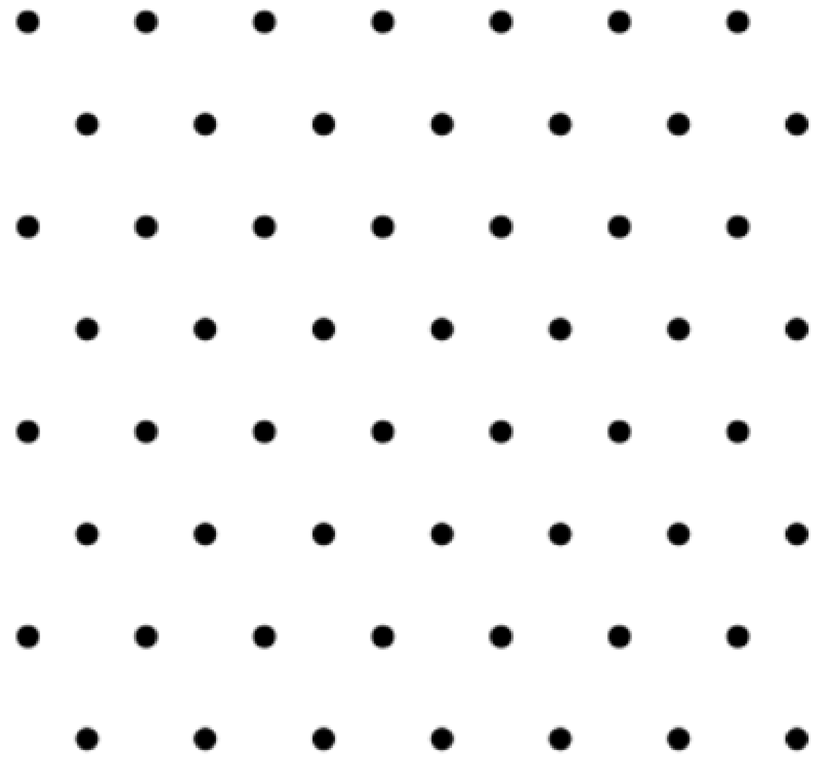

Stable Roommates

[Gale and Shapley, 1962]

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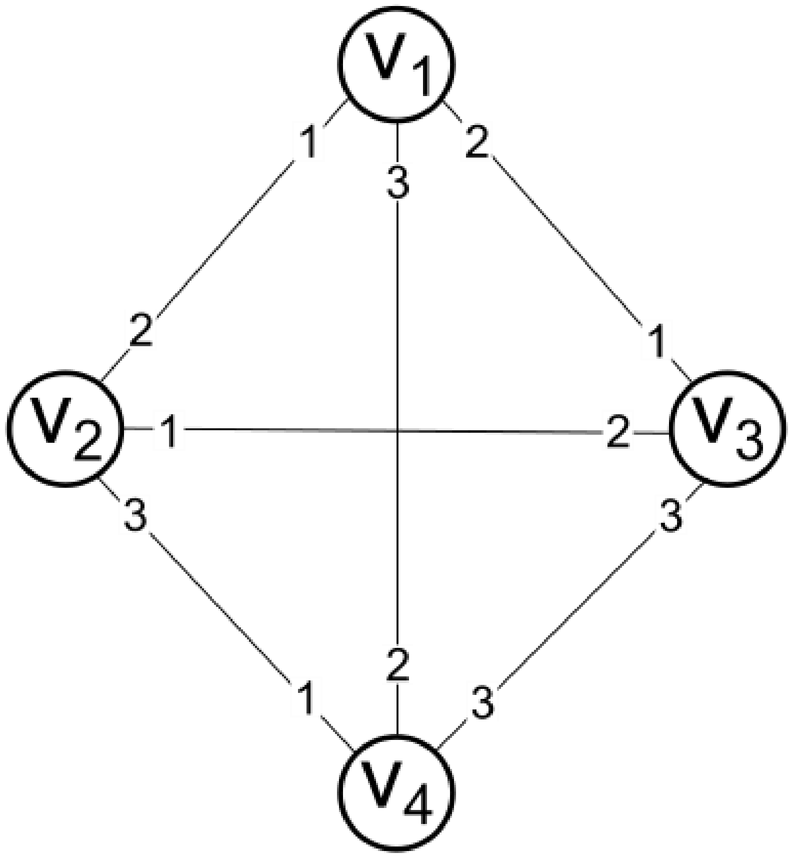

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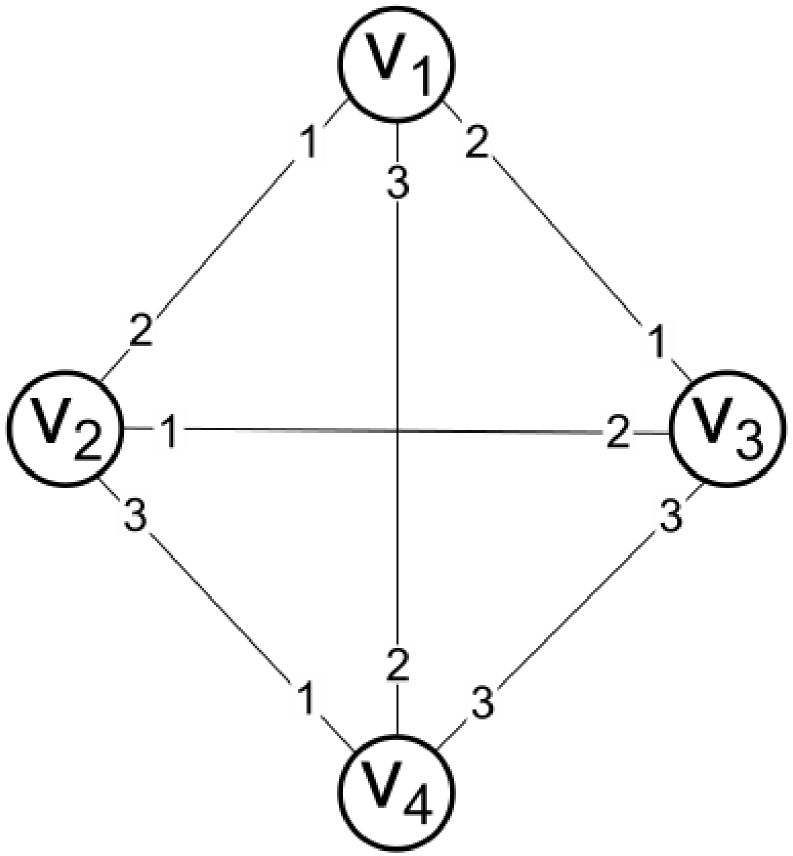

A matching is stable is there is no blocking pair of vertices that prefer each other over their assigned partners ("self-partnered" if unmatched).

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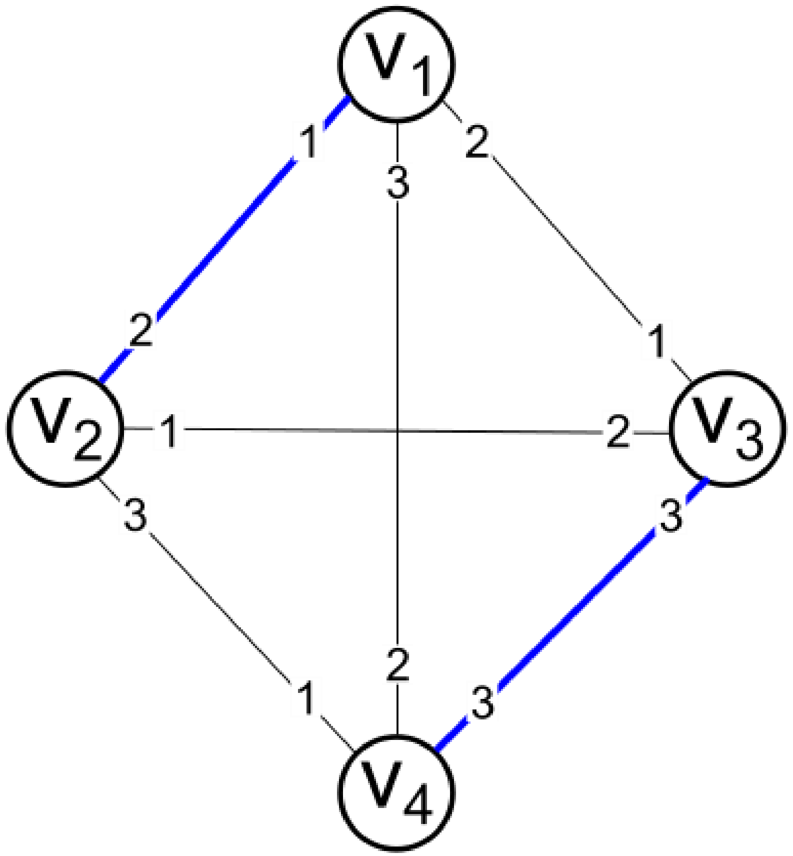

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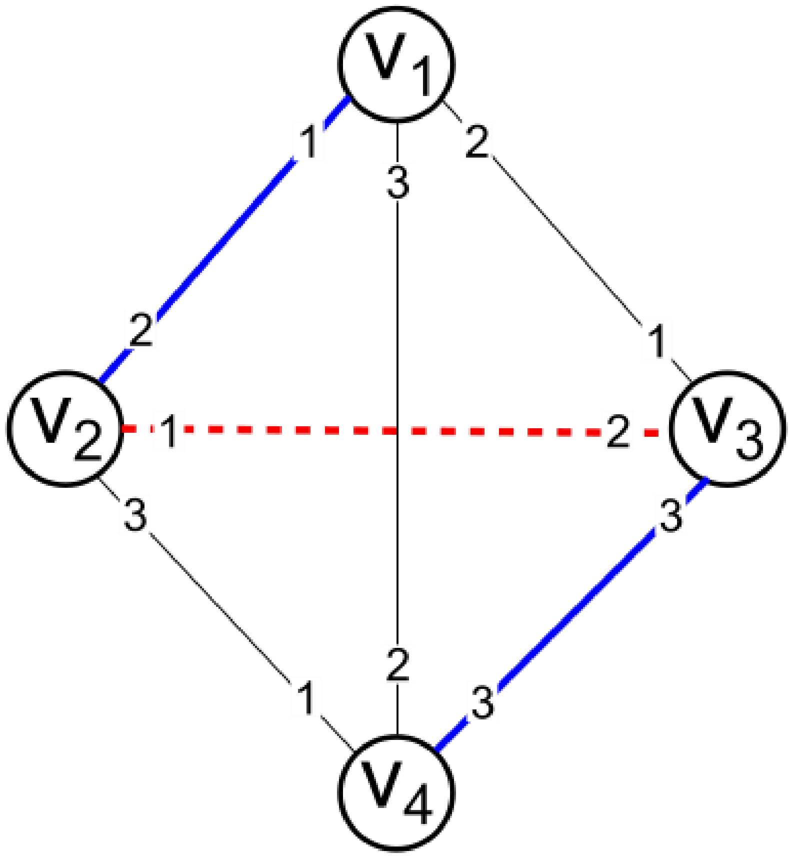

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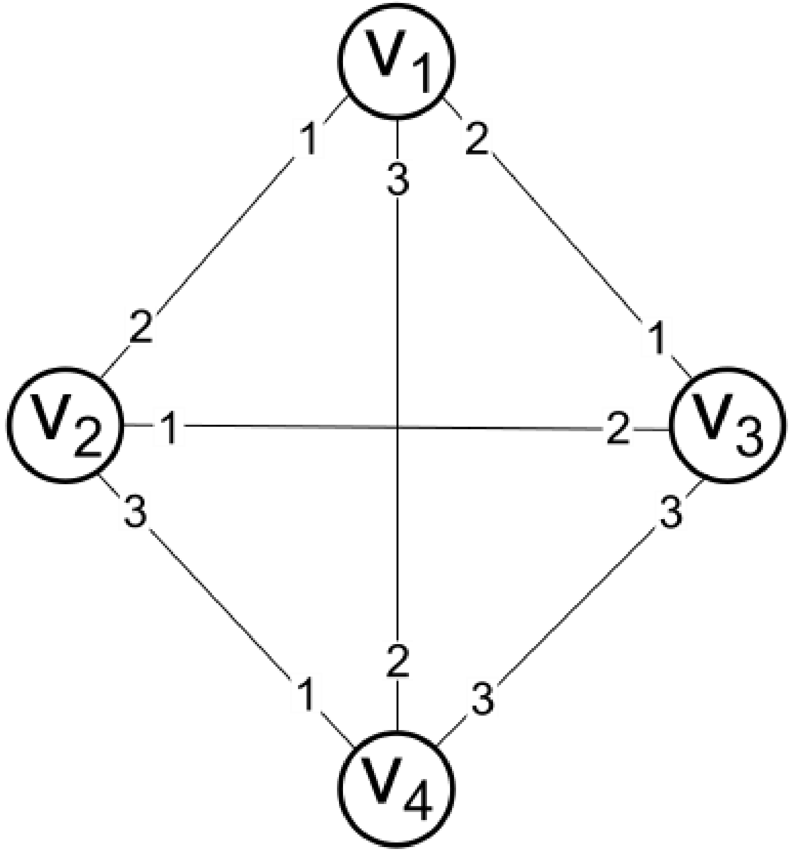

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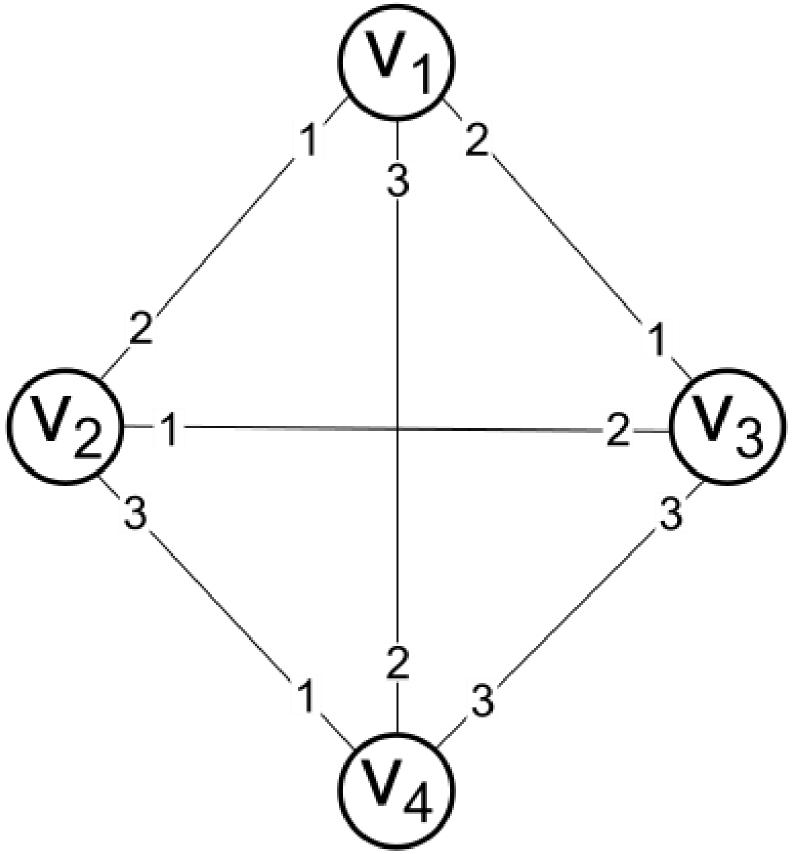

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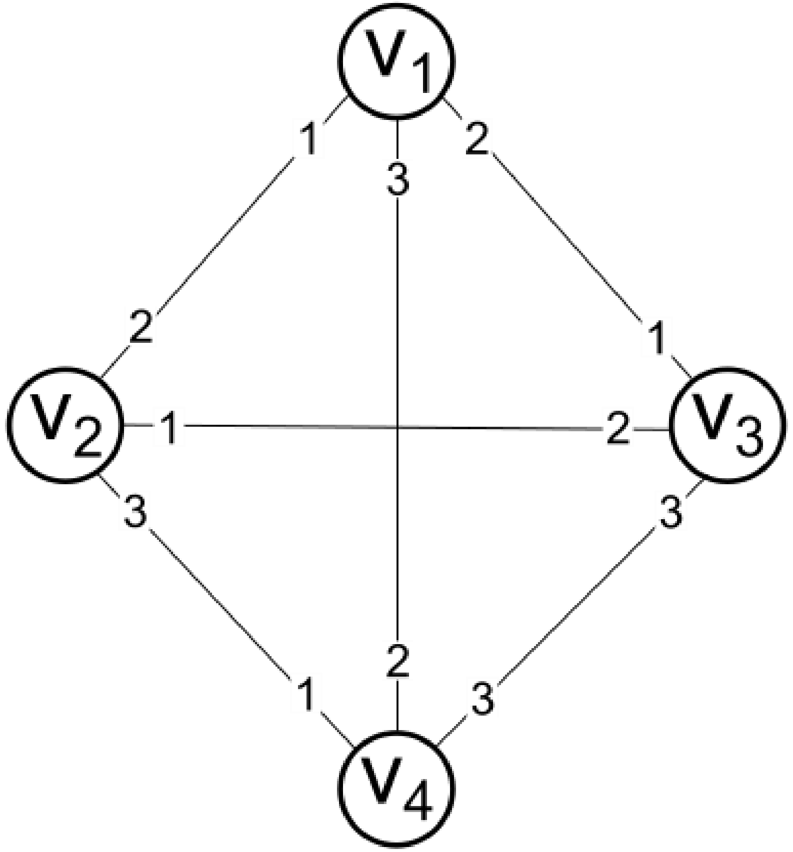

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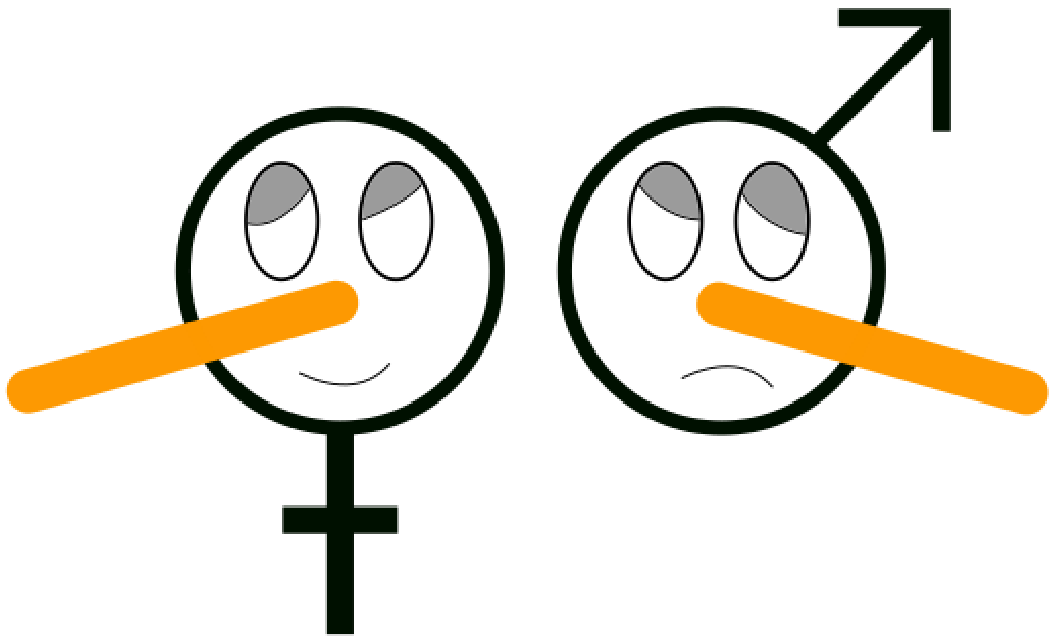
Whoever is matched with v_4 will block with one of the other two agents.

Next Time

Incentives in the Stable Matching Problem



References

- Stability and the Deferred Acceptance Algorithm

  David Gale and Lloyd Shapley
  "*College Admissions and the Stability of Marriage*"
  American Mathematical Monthly,69(1), 1962 pg 9-15
  https://www.jstor.org/stable/2312726

- Structure of the Set of Stable Matchings

  Alvin Roth and Marilda Sotomayor
  "*Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*"
  Econometric Society Monograph Series, 1990